SOLUTIONS OF SPECIAL ASYMPTOTICS TO THE EINSTEIN CONSTRAINT EQUATIONS

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Abstract. We construct solutions of specific asymptotics to the Einstein constraint equations using a cut–off technique. Moreover, we give various examples of vacuum asymptotically flat manifolds whose center of mass and angular momentum fail to exist.

1. Introduction

Let M be a three–dimensional manifold. Let g be a Riemannian metric and K be a symmetric $(0,2)$–tensor on M. The Einstein constraint equations are

$$R_g - |K|_g^2 + (\mathrm{tr}_g K)^2 = 2\mu,$$
$$\mathrm{div}_g(K - (\mathrm{tr}_g K)g) = J,$$

where μ and J are energy density and momentum density respectively. The triple (M,g,K) is called an initial data set if it satisfies the above constraint equations. It is called a *vacuum* initial data set if additionally $\mu = 0$ and $J = 0$. In general relativity, the constraint equations are from the Gauss and Codazzi equations for a hypersurface M in spacetime with the induced metric g and the induced second fundamental form K.

An initial data set (M,g,K) is called *asymptotically flat* at the decay rate q if, outside a compact set, M is diffeomorphic to $\mathbb{R}^3 \setminus B_1$ and if there exists an asymptotically flat chart $\{x\}$ so that, for some $q > 1/2$,

$$g_{ij}(x) = \delta_{ij} + O_2(r^{-q}), \quad K_{ij}(x) = O_1(r^{-1-q})$$

and

$$\mu(x) = O(r^{-2-2q}), \quad J(x) = O(r^{-2-2q}),$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and the subscript in the big O notation indicates the corresponding decay rate on successive derivatives, e.g. $f = O_1(r^{-q})$ means that there is a constant C uniformly in r so that $|f| \le Cr^{-q}$ and $|\partial f| \le Cr^{-1-q}$.

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We introduce the momentum tensor $\pi = K - (\mathrm{tr}_g K)g$, and define the constraint map

$$\Phi(g,\pi) = \left(R_g - |\pi|_g^2 + \frac{1}{2}(\mathrm{tr}_g \pi)^2, \mathrm{div}_g \pi \right).$$

Then the Einstein constraint equations take the form $\Phi(g,\pi) = (2\mu, J)$.

In the asymptotically flat chart, the following physical quantities are defined as limits of surface integrals over Euclidean spheres with the standard surface measure on $\{r = \rho\}$:

$$m = \frac{1}{16\pi} \lim_{\rho\to\infty} \int_{r=\rho} \sum_{i,j} (g_{ij,i} - g_{ii,j}) \frac{x_j}{r}\, dS, \tag{1.1}$$

$$\mathcal{P}_i = \frac{1}{8\pi} \lim_{\rho\to\infty} \int_{r=\rho} \sum_j \pi_{ij} \frac{x_j}{r}\, dS, \tag{1.2}$$

$$\mathcal{C}_l = \frac{1}{16\pi m} \lim_{\rho\to\infty} \int_{r=\rho} \left[x_l \sum_{i,j} (g_{ij,i} - g_{ii,j}) \frac{x_j}{r} - \sum_i \left(g_{il} \frac{x_i}{r} - g_{ii} \frac{x_l}{r} \right) \right] dS, \tag{1.3}$$

$$\mathcal{J}_i = \frac{1}{8\pi} \lim_{\rho\to\infty} \int_{r=\rho} \sum_{j,k} \pi_{jk} Y_{(i)}^j \frac{x_k}{r}\, dS, \tag{1.4}$$

where $Y_{(i)}$ are the rotation vector fields, e.g. $Y_{(1)} = x_3\partial_2 - x_2\partial_3$. These integrals correspond to the ADM mass m, linear momentum $\mathcal{P}$, center of mass $\mathcal{C}$, and angular momentum $\mathcal{J}$.

It is well–known that the ADM mass and linear momentum of an asymptotically flat manifold are well–defined [1, 3]. However, center of mass and angular momentum may not be well–defined unless some extra condition (for example, the RT condition below) is imposed.

Definition 1.1. *(M, g, K) is asymptotically flat satisfying the Regge-Teitelboim condition (the RT condition) if (M, g, K) is asymptotically flat, and g, K satisfy these asymptotically even/odd conditions:*

$$g_{ij}^{odd}(x) = O_2(r^{-1-q}), \quad K_{ij}^{even}(x) = O_1(r^{-2-q})$$

and

$$\mu^{odd}(x) = O(r^{-3-2q}), \quad J^{odd}(x) = O(r^{-3-2q}),$$

where $f^{odd}(x) = f(x) - f(-x)$ and $f^{even}(x) = f(x) + f(-x)$ are respectively the even and odd parts of f with respect to the fixed asymptotically flat chart $\{x\}$.

Assuming $m \neq 0$, the center of mass and angular momentum are well–defined for asymptotically flat manifolds with the RT condition

[2, 5, 6]. Moreover, all known exact solutions to the constraint equations satisfy the RT condition. In particular, two well–known families of solutions to the vacuum constraint equations are Schwarzschild and Kerr, which satisfy the RT condition. It was not clear whether the vacuum asymptotically flat manifolds *without* the RT condition do exist, because one may tend to think the asymptotics of the solutions to the vacuum constraint equations are rigid. We show that the asymptotics are not rigid. Indeed, we can construct solutions without the RT condition. Using a cut–off technique of Corvino and Schoen [4], we have the following theorem.

Assume that σ, τ are symmetric $(0,2)$–tensors defined outside a compact set in $\mathbb{R}^3$. Assume that σ_{ij}, τ_{ij} are components of σ, τ with respect to the standard Euclidean coordinate chart $\{x\}$.

Theorem 1. *Assume that σ and τ satisfy the linearized constraint equations outside a compact set in $\mathbb{R}^3$, i.e.*

$$\sum_{i,j}(\sigma_{ij,ij} - \sigma_{ii,jj}) = 0, \tag{1.5}$$

$$\sum_{i}\tau_{ij,i} = 0, \quad \text{for } j = 1,2,3. \tag{1.6}$$

Furthermore, assume that $\sigma_{ij}(x) = O_2(r^{-q}), \tau_{ij}(x) = O_1(r^{-1-q})$ for $q \in (1/2, 1)$. Then given any asymptotically flat initial data set (g, π) at the decay rate greater than or equal to q and $\Phi(g,\pi) = (2\mu, J)$, there exists an asymptotically flat initial data set $(\overline{g}, \overline{\pi})$ with $\Phi(\overline{g}, \overline{\pi}) = (2\mu, J)$ so that, for some constants A and $B_i, i = 1,2,3,$

$$\overline{g}_{ij} = \left(1 + \frac{A}{r}\right)\delta_{ij} + \sigma_{ij} + O_2(r^{-1-q}), \tag{1.7}$$

$$\overline{\pi}_{ij} = \tau_{ij} + \frac{1}{r^3}\left[-B_i x_j - B_j x_i + \sum_k (B_k x_k)\delta_{ij}\right] + O_1(r^{-2-q}), \tag{1.8}$$

and (g,π) and $(\overline{g}, \overline{\pi})$ are close (in the sense of weighted Sobolev spaces).

In order to construct solutions of special asymptotics, one has to find explicit σ and τ satisfying (1.5) and (1.6). In section 3, we give examples of σ and τ. Therefore, we can construct families of asymptotically flat manifolds without the RT condition and show that their enter of mass and angular momentum (1.3) and (1.4) diverge (Corollaries 3.4, 3.7, and 3.8). It is desirable to weaken the RT condition in order to define center of mass and angular momentum. The examples in section 3 may help us to understand these physical quantities.

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2. Constructing Solutions of Specific Asymptotics

To prove Theorem 1, we introduce the weighted Sobolev spaces.

Definition 2.1 (Weighted Sobolev Spaces)**.** *For an integer $k \geq 0$, a real number $p \geq 0$, and a real number q, we say $f \in W^{k,p}_{-q}$ if*

$$\|f\|_{W^{k,p}_{-q}} \equiv \left(\int_M \sum_{|\alpha| \leq k} \left(\left| D^\alpha f \right| \xi^{|\alpha|+q} \right)^p \xi^{-3} \, dvol_g \right)^{\frac{1}{p}} < \infty,$$

where α is a multi-index and ξ is a continuous function with $\xi = |x|$ when the asymptotically flat chart is defined. When $p = \infty$,

$$\|f\|_{W^{k,\infty}_{-q}} = \sum_{|\alpha| \leq k} ess \sup_M |D^\alpha f| \xi^{|\alpha|+q}.$$

Then we can weaken the definition of asymptotically flat manifolds and define (g, π) to be asymptotically flat at the decay rate q if

$$(g - \delta, \pi) \in W^{2,p}_{-q} \times W^{1,p}_{-1-q}$$

and $\Phi(g, \pi) = (\mu, J)$ with

$$(\mu, J) \in W^{0,p}_{-2-2q} \times W^{0,p}_{-2-2q}.$$

In the proof, we assume that $p > 3/2$ and $q \in (1/2, 1)$.

Proof of Theorem 1. Let $\{\phi_k\}$ be a sequence of smooth cut–off functions:

$$\phi_k(x) = \begin{cases} 1 & \text{in } B_k, \\ \text{between 0 and 1} & \text{in } B_{2k} \setminus B_k, \\ 0 & \text{outside } B_{2k}. \end{cases}$$

Also $\{\phi_k\}$ is chosen so that ϕ_k is radial and $|\partial \phi_k| \leq C/k$ and $|\partial^2 \phi_k| \leq C/k^2$ for some constant C independent of k.

Let (g, π) be a given asymptotically flat manifold at the decay rate q. Using the cut–off technique, we consider

$$\hat{g}_k = \phi_k g + (1 - \phi_k)(\delta + \sigma), \tag{2.1}$$

$$\hat{\pi}_k = \phi_k \pi + (1 - \phi_k)\tau. \tag{2.2}$$

For the moment, we work on a fixed k and suppress the subscript k when it is clear from context.

By (1.5), (1.6), and the properties of ϕ_k,

$$\Phi(\hat{g}, \hat{\pi}) = (\hat{\mu}, \hat{J}),$$

where $(\hat{\mu}, \hat{J}) = (\mu, J)$ in B_k and $(\hat{\mu}, \hat{J}) = (O(r^{-2-2q}), O(r^{-2-2q}))$ outside B_{2k} because of (1.5) and (1.6).

In order to fully solve the constraint equations with the given μ and J, we consider a function u and a vector field X so that

$$\overline{g} = u^4 \hat{g}, \tag{2.3}$$

$$\overline{\pi} = u^2(\hat{\pi} + \mathcal{L}_{\hat{g}} X), \tag{2.4}$$

where $\mathcal{L}_g X$ denotes the modified Lie derivative $\mathcal{L}_g X = L_g X - (\mathrm{div}_g X) g$ for a Riemannian metric g.

claim: There exists (u_k, X_k) with $(u_k - 1, X_k) \in W^{2,p}_{-q} \times W^{2,p}_{-q}$, and $(h_k, w_k) \in W^{2,p}_{-q} \times W^{1,p}_{-1-q}$ with the compact supports such that

$$\Phi(\overline{g}_k + h_k, \overline{\pi}_k + w_k) = (\mu, J) \tag{2.5}$$

for k large.

Proof of the claim. The proof is similar to the argument by Corvino and Schoen [4, Theorem 1]. The only (minor) difference is that they work on *vacuum* initial data sets, i.e. $\mu = 0$ and $J = 0$. To see the argument works for general μ and J, we only highlight the different part of the argument.

Let $T_k : (1 + W^{2,p}_{-q}) \times W^{2,p}_{-q} \to W^{0,p}_{-2-q} \times W^{0,p}_{-2-q}$ be defined by

$$T_k(u, X) = \Phi(u^4 \hat{g}_k, u^2(\hat{\pi}_k + \mathcal{L}_{\hat{g}_k} X))$$

be a sequence of operators. The linearization $D(T_k)_{(1,0)}$ is a Fredholm operator with index 0 for k large. Using the surjectivity of $D\Phi_{(\hat{g}_k, \hat{\pi}_k)}$, one can define

$$\overline{T}_k\left((u, X), (h, w)\right) = \Phi\left(u^4 \hat{g}_k + h, u^2(\hat{\pi}_k + \mathcal{L}_{\hat{g}_k} X) + w\right)$$

where (h, w) is chosen so that $D\Phi_{(\hat{g}_k, \hat{\pi}_k)}(h, w)$ is in the cokernel of $D(T_k)_{(1,0)}$. Therefore, $D(\overline{T}_k)_{((1,0),(0,0))}$ is an isomorphism for each large k by construction. Then by applying the inverse function theorem, $\overline{T}_k$ is a diffeomorphism from a neighborhood of $((1,0),(0,0))$ to a *fixed* (independent of k) neighborhood of $\overline{T}_k((1,0),(0,0))$ when k large. Then the image contains (μ, J) when k large, and hence, we can find the sequence of initial data sets so that (2.5) holds. □

It remains to check that $(\overline{g}, \overline{\pi})$ has the desired asymptotics (1.7) and (1.8). Fix k. Outside a large compact set, if we denote $\widetilde{g} = \delta + \sigma$ and $\mathcal{L} = \mathcal{L}_{\widetilde{g}}$, we have, by (2.1), (2.2), (2.3), and (2.4),

$$\overline{g} = u^4 \widetilde{g},$$

$$\overline{\pi} = u^2(\tau + \mathcal{L} X).$$

Because $(\overline{g}, \overline{\pi})$ satisfies the constraint equations, we can derive the differential equation for u:

$$\begin{aligned}
&-8\Delta_{\widetilde{g}}u + \left[R_{\widetilde{g}} - |\tau|^2_{\widetilde{g}} + \frac{1}{2}\left(\mathrm{tr}_{\widetilde{g}}\tau\right)^2\right]u\\
&+ u\left[\mathrm{tr}_{\widetilde{g}}(\tau)\mathrm{tr}_{\widetilde{g}}(\mathcal{L}X) + \frac{1}{2}(\mathrm{tr}_{\widetilde{g}}(\mathcal{L}X))^2\right]\\
&- u\left[\sum_{i,j,k,l}\widetilde{g}^{ij}\widetilde{g}^{kl}2\tau_{ik}(\mathcal{L}X)_{jl} + |\mathcal{L}X|^2_{\widetilde{g}}\right] = 2u^5\mu,
\end{aligned}$$

and the differential equations for X_i:

$$\begin{aligned}
\mathrm{div}_{\overline{g}}\overline{\pi} &= u^{-4}\widetilde{g}^{jk}\left[u^2(\tau+\mathcal{L}X)_{ij;k} + 2uu_{,k}(\tau+\mathcal{L}X)_{ij}\right]\\
&= u^{-2}\mathrm{div}_{\widetilde{g}}\tau + u^{-2}\mathrm{div}_{\widetilde{g}}(\mathcal{L}X) + 2u^{-3}u_{,k}\widetilde{g}^{jk}(\tau+\mathcal{L}X)_{ij}\\
&= J.
\end{aligned}$$

Then, for r large, u and $\{X_i\}_{i=1}^3$ satisfy the differential equations of the standard Laplacian Δ:

$$\Delta u = O(r^{-2-2q}), \quad \text{and} \quad \Delta X_i = O(r^{-2-2q}).$$

Therefore, u and X_i are harmonic up to lower terms; that is,

$$\begin{aligned}
u &= 1 + \frac{A}{4r} + O(r^{-2q}),\\
X_i &= \frac{B_i}{r} + O(r^{-2q}),
\end{aligned}$$

for some constants A and $\{B_i\}_{i=1}^3$. Then (1.7) and (1.8) follow. □

3. Solutions of Special Asymptotics

In order to construct solutions of explicit asymptotics, the key is to solve σ and τ in (1.5) and (1.6). In this section, we show some special examples of σ and τ and use Theorem 1 to construct asymptotically flat initial data sets which do *not* satisfy the RT condition. In subsection 3.1, we discuss a family of solutions at decay rate exactly equal to one. In subsection 3.2, we discuss another family of solutions whose decay rate is less than 1. Moreover, we show that the center of mass and angular momentum of these examples do not exist.

3.1. **Solutions at the decay rate equal to 1.** Let σ and τ be

$$\sigma_{ij} = \frac{\alpha}{r}\left(\frac{x_i x_j}{r^2} - \frac{1}{2}\delta_{ij}\right), \tag{3.1}$$

$$\tau_{ij} = \frac{\beta}{r^2}\frac{x_i x_j}{r^2}, \tag{3.2}$$

where α and β are arbitrary C^2 functions defined over the unit sphere S^2. Because α and β are independent of r, by direct computations, we have the following lemma:

Lemma 3.1. *For any $\alpha, \beta \in C^2(S^2)$, σ and τ satisfy the linearized constraint equations* (1.5) *and* (1.6).

Proposition 3.2. *For any $\alpha, \beta \in C^2(S^2)$, there exists a vacuum initial data set $(\overline{g}, \overline{\pi})$ with the following asymptotics:*

$$\overline{g}_{ij} = \left(1 + \frac{A}{r}\right)\delta_{ij} + \frac{\alpha}{r}\left(\frac{x_i x_j}{r^2} - \frac{1}{2}\delta_{ij}\right) + O_2(r^{-1-q}), \tag{3.3}$$

$$\overline{\pi}_{ij} = \frac{\beta}{r^2}\frac{x_i x_j}{r^2} + \frac{1}{r^3}\left[-B_i x_j - B_j x_i + \sum_k (B_k x_k)\delta_{ij}\right] + O_1(r^{-2-q}), \tag{3.4}$$

Proof. The proposition follows by choosing $(g, \pi) = (\delta, 0)$ and (σ, τ) as (3.1) and (3.2) in Theorem 1. □

Remark. *Asymptotically flat manifolds of the above asymptotics have been discovered by Beig and Ó Murchadha* [2]. *They showed that $(\overline{g}, \overline{\pi})$ satisfies the vacuum constraint equations up to leading order terms by direct computations. Here, we provide a more rigorous treatment and prove that $(\overline{g}, \overline{\pi})$ indeed satisfies the vacuum constraint equations.*

Examples of divergent angular momentum. We can construct the asymptotically flat manifolds whose angular momentum with respect to a rotation vector field Y diverges.

Let $(\overline{g}, \overline{\pi})$ be an asymptotically flat manifold of the asymptotics (3.3) and (3.4). Fix ρ_0 and let $A_\rho = \{x \in \mathbb{R}^3 : \rho_0 \le |x| \le \rho\}$.

Lemma 3.3. *For any $\alpha, \beta \in C^2(S)$,*

$$\int_{\partial A_\rho} \sum_{i,j} \overline{\pi}_{ij} Y^i \frac{x_j}{r}\, dS = \frac{1}{4}\int_{A_\rho} \sum_p \frac{\alpha_{,p}\beta}{r^3} Y^p\, dx - \int_{A_\rho} \frac{\alpha}{r^4} \sum_{i,j} Y^i_{,j} B_j x_i\, dx$$
$$+ \int_{A_\rho}\left(-\frac{3}{4r^3}\sum_{i,p}(B_i x_i \alpha_{,p} Y^p) - \frac{\alpha}{r^4}\sum_p B_p Y^p\right) dx + O(1), \tag{3.5}$$

where and in the following $O(1)$ denotes the term bounded uniformly in ρ.

Proof. We compute the angular momentum (3.5) over the annulus. Notice that because $\overline{g} = \delta + O(r^{-1})$

$$\int_{\partial A_\rho} \sum_{i,j} \overline{\pi}_{ij} Y^i \frac{x^j}{r}\, dS = \int_{\partial A_\rho} \sum_{i,j} \overline{\pi}_{ij} Y^i \nu^j \, dS_{\overline{g}} + O(1).$$

Then by the divergence theorem,

$$\begin{aligned}\int_{\partial A_\rho} \sum_{i,j} \overline{\pi}_{ij} Y^i \nu^j \, dS_{\overline{g}} &= \int_{A_\rho} \sum_{i,j,k} \overline{g}^{jk} \overline{\pi}_{ki} Y^i_{;j} \, dvol_{\overline{g}} \\ &= \int_{A_\rho} \sum_{i,j,k} \overline{g}^{jk} \overline{\pi}_{ki} Y^i_{,j} \, dvol_{\overline{g}} + \int_{A_\rho} \sum_{i,j,k,p} \overline{g}^{jk} \overline{\pi}_{ki} Y^p \overline{\Gamma}^i_{jp} \, dvol_{\overline{g}}. \qquad (3.6)\end{aligned}$$

In the first equality, we use the constraint equation $\mathrm{div}_{\overline{g}} \overline{\pi} = 0$. Then because Y is Killing (with respect to the Euclidean metric),

$$\int_{A_\rho} \sum_{i,j,k} \overline{g}^{jk} \overline{\pi}_{ki} Y^i_{,j} \, dvol_{\overline{g}} = \int_{A_\rho} \sum_{i,j,k} (\overline{g}^{jk} - \delta^{jk}) \overline{\pi}_{ki} Y^i_{,j} \, dvol_{\overline{g}}$$

By (1.7) and (1.8), the above integral is equal to

$$\begin{aligned}&\int_{A_\rho} \left[-\sum_{i,j,k} \sigma_{jk} \tau_{ki} Y^i_{,j} - \frac{A}{r} \sum_{i,j} \tau_{ij} Y^i_{,j} \right] dx \\ &- \int_{A_\rho} \frac{1}{r^3} \sum_{i,j,k} \sigma_{jk} Y^i_{,j} \left(-B_k x_i - B_i x_k + \sum_l B_l x_l \delta_{ik} \right) dx + O(1). \qquad (3.7)\end{aligned}$$

Then by (3.1) and (3.2), the first line vanishes:

$$\begin{aligned}\sum_{i,j,k} \sigma_{jk} \tau_{ki} Y^i_{,j} &= \frac{\alpha\beta}{r^3} \sum_{i,j,k} \left(\frac{x_j x_k}{r^2} - \frac{1}{2} \delta_{jk} \right) \frac{x_i x_k}{r^2} Y^i_{,j} = \frac{\alpha\beta}{2r^4} \sum_{i,j} \frac{x_i x_j}{r} Y^i_{,j} \\ &= \frac{\alpha\beta}{2r^4} \sum_i x_i \frac{\partial Y^i}{\partial r} = \frac{\alpha\beta}{2r^4} \frac{\partial}{\partial r} \left(\sum_i x_i Y^i \right) = 0,\end{aligned}$$

where we use that Y is a rotation vector field and hence $\sum_i x_i Y^i = 0$. The other term can be computed similarly. The second line in (3.7) is

$$-\int_{A_\rho} \frac{\alpha}{r^4} \sum_{i,j} Y^i_{,j} B_j x_i \, dx + O(1).$$

Because $\overline{g} = \delta + O(r^{-1})$, the second integral in (3.6) is

$$\int_{A_\rho} \sum_{i,j,p} \overline{g}^{jk} \overline{\pi}_{ki} Y^p \overline{\Gamma}^i_{jp} \, dvol_{\overline{g}} = \frac{1}{2} \int_{A_\rho} \sum_{i,j,p} \overline{\pi}_{ij} Y^p \overline{g}_{ij,p} \, dx + O(1)$$

$$= \frac{1}{2} \int_{A_\rho} \sum_{i,j,p} \frac{1}{r^3} \left[-2B^i x^j + \sum_k (B_k x_k) \delta_{ij} \right] Y^p \sigma_{ij,p} \, dx$$

$$+ \frac{1}{2} \int_{A_\rho} \sum_{i,j,p} \tau_{ij} Y^p \sigma_{ij,p} \, dx + O(1).$$

Then the lemma follows by substituting σ and τ by (3.1) and (3.2), □

Corollary 3.4. *If we choose*

$$\alpha = \frac{x_1^2}{r^2}, \quad \beta = \frac{x_1 x_3}{r^2}, \quad \textit{and} \quad Y = x_3 \partial_1 - x_1 \partial_3,$$

the integral of the angular momentum with respect to Y diverges.

Proof. By Lemma 3.3 and the straightforward computations,

$$\frac{1}{4} \int_{A_\rho} \sum_p \frac{\alpha_{,p} \beta}{r^3} Y^p \, dx = \frac{1}{2} \int_{A_\rho} \frac{x_1^2 x_3^2}{r^7} \, dx \to \infty \quad \text{as } \rho \to \infty.$$

The other terms in (3.5) vanish because α is an even function and Y is odd. □

Remark. *If α is a constant, then the angular momentum exists no matter what choices of β are made.*

Examples of divergent center of mass. We construct an explicit example of an asymptotically flat manifold whose center of mass diverges.

Lemma 3.5. *For any asymptotically flat Riemannian metric $g = \delta + O_2(r^{-1})$, the scalar curvature has the asymptotics:*

$$R_g(x) = \sum_{i,j} (g_{ij,ij} - g_{ii,jj}) + E_g + O(r^{-5}), \quad \textit{when } r \textit{ large}, \qquad (3.8)$$

where

$$E_g = \sum_{i,j,l} \left(-g_{jl,j} g_{il,i} + \frac{3}{4} g_{ij,l} g_{ij,l} - \frac{1}{4} g_{jj,l} g_{ii,l} - \frac{1}{2} g_{ij,l} g_{il,j} + g_{jl,j} g_{ii,l} \right).$$

Proof. For any Riemannian metric g, over a coordinate chart, we have

$$R_g = \sum_{i,j} g(\nabla_{\partial_j} \nabla_{\partial_i} \partial_i - \nabla_{\partial_i} \nabla_{\partial_j} \partial_i, \partial_j)$$

$$= \sum_{i,j,p} (\Gamma^p_{ii,j} g_{pj} - \Gamma^p_{ij,i} g_{pj}) + \sum_{i,j,p,m} (\Gamma^p_{ii} \Gamma^m_{jp} g_{mj} - \Gamma^p_{ij} \Gamma^m_{ip} g_{mj}).$$

Using the property that $g = \delta + O_2(r^{-1})$ and by direct computations, we have the following asymptotics:

$$\sum_{i,j,p} \Gamma^p_{ii,j} g_{pj} = \sum_{i,j,l} \left[-\frac{1}{2} g_{jl,j}(2g_{il,i} - g_{ii,l}) + \frac{1}{2}(2g_{ij,ij} - g_{ii,jj}) \right] + O(r^{-5}),$$

$$\sum_{i,j,p} \Gamma^p_{ij,i} g_{pj} = \sum_{i,j,l} \left[-\frac{1}{2} g_{jl,i} g_{jl,i} + \frac{1}{2} g_{jj,ii} \right] + O(r^{-5}),$$

$$\sum_{i,j,p,m} \Gamma^p_{ii} \Gamma^m_{jp} g_{mj} = \sum_{i,j,l} \frac{1}{4} g_{jj,l}(2g_{il,i} - g_{ii,l}) + O(r^{-5}),$$

$$\sum_{i,j,p,m} \Gamma^p_{ij} \Gamma^m_{ip} g_{mj} = \sum_{i,j,l} \frac{1}{4} g_{ij,l}(2g_{il,j} - g_{ij,l}) + O(r^{-5}).$$

Combining above identities, we derive (3.8). □

Let $(\overline{g}, \overline{\pi})$ be an asymptotically flat manifold of the asymptotics (1.7) and (1.8). If we let $(g, \pi) = (\delta, 0)$ and $\tau = 0$, from the proof of Theorem 1, we have $\overline{\pi} = 0$ and hence $\overline{g}$ satisfies the constraint equation $R_{\overline{g}} = 0$.

Proposition 3.6. *Let $\overline{g}$ satisfy* (3.3) *with $R_{\overline{g}} = 0$. Its center of mass is equal to, for $l = 1, 2, 3$,*

$$\mathcal{C}_l = \frac{1}{16\pi m} \lim_{\rho \to \infty} \int_{A_\rho} x_l \left[-\frac{29|\nabla \alpha|^2}{16 r^2} + \frac{A\alpha}{2r^4} + \frac{\alpha^2}{8r^4} \right] dx + C_l, \qquad (3.9)$$

for some constant C_l.

Proof. Let $A_\rho = \{x \in \mathbb{R}^3 : \rho_0 \le |x| \le \rho\}$ for some fixed ρ_0. Then by the divergence theorem

$$\int_{\partial A_\rho} \left[x_l \sum_{i,j} (\overline{g}_{ij,i} - \overline{g}_{ii,j}) \frac{x_j}{r} - \sum_i \left(\overline{g}_{il} \frac{x_i}{r} - \overline{g}_{ii} \frac{x_l}{r} \right) \right] dS$$
$$= \int_{A_\rho} x_l \sum_{i,j} (\overline{g}_{ij,ij} - \overline{g}_{ii,jj})\, dx.$$

Using the identity (3.8),

$$x^l \sum_{i,j} (\overline{g}_{ij,ij} - \overline{g}_{ii,jj}) = x^l R_{\overline{g}} - x^l E_{\overline{g}} + O(r^{-4}).$$

The first term of the scalar curvature vanishes, and the third term is integrable over A_ρ. It remains to compute $\int_{A_\rho} x^l E_{\overline{g}}\, dx$. By (3.3) and direct computations, the following identities hold up to terms of order

$O(r^{-5})$:

$$\sum_{i,j,l} \overline{g}_{jl,j}\overline{g}_{il,i} = \frac{A^2}{r^4} + \frac{|\nabla\alpha|^2}{4r^2} + \frac{1}{r^4}\left(\frac{9}{4}\alpha^2 - 3A\alpha\right),$$

$$\sum_{i,j,l} \overline{g}_{ij,l}^2 = \frac{3A^2}{r^4} + \frac{3|\nabla\alpha|^2}{r^4} + \frac{19\alpha^2}{4r^4} - \frac{A\alpha}{r^4},$$

$$\sum_{i,j,l} \overline{g}_{jj,l}\overline{g}_{ii,l} = \frac{|\nabla\alpha|^2}{4r^2} + \frac{1}{r^4}\left(3A - \frac{1}{2}\alpha\right)^2,$$

$$\sum_{i,j,l} \overline{g}_{ij,l}\overline{g}_{il,j} = \frac{A^2}{r^4} + \frac{|\nabla\alpha|^2}{4r^2} + \frac{17\alpha^2}{4r^4} - \frac{3A\alpha}{r^4},$$

$$\sum_{i,j,l} \overline{g}_{jl,j}\overline{g}_{ii,l} = \frac{|\nabla\alpha|^2}{4r^2} + \frac{1}{r^4}\left(A - \frac{3}{2}\alpha\right)\left(3A - \frac{1}{2}\alpha\right).$$

Therefor, (3.9) follows by combining the above identities. □

Corollary 3.7. *If we choose*

$$\alpha = \frac{x_1}{r},$$

then the first component of the center of mass $\mathcal{C}_1$ is infinity.

3.2. **Solutions at the decay rate less than 1.** We consider another family of τ satisfying (1.6). Let u be any C^2 function on $\mathbb{R}^3$. Let

$$\tau_{ij} = (|\nabla u|^2 + u\Delta u)\delta_{ij} - (u_i u_j + u u_{ij}).$$

By direct computations,

$$\sum_i \tau_{ij,i} = 0 \quad \text{for } j = 1, 2, 3.$$

We can choose, for example, $u = \log r$. Then

$$\tau_{ij} = \frac{1}{r^2}\delta_{ij} + \frac{x_i x_j}{r^4}(2\log r - 1). \tag{3.10}$$

Notice that $\tau_{ij} \neq O_1(r^{-2})$ because the logarithmic term. More generally, if we let $u = r^{(1-q)/2}$ for $q < 1$, $\tau = O_1(r^{-1-q})$.

Choosing this particular τ from (3.10), we have another example of a vacuum asymptotically flat manifold whose angular momentum is not defined.

Corollary 3.8. *Let $(\overline{g}, \overline{\pi})$ satisfy* (1.7) *and* (1.8) *where σ and τ have the expression* (3.1) *and* (3.10). *Then if*

$$\alpha = \tan^{-1}\left(\frac{x_2}{x_1}\right),$$

the angular momentum with respect to $Y = -x_2\partial_1 + x_1\partial_2$ *diverges.*

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