ON THE CENTER OF MASS AND CONSTANT MEAN CURVATURE SURFACES OF ASYMPTOTICALLY FLAT INITIAL DATA SETS

LAN-HSUAN HUANG

Contents

1. Introduction 1
2. Uniqueness of Embedded CMC Surfaces 5
3. Stable CMC Surfaces 8
3.1. Variational formulas 8
3.2. Uniqueness of stable CMC surfaces 11
4. Existence of CMC surfaces in asymptotically flat initial data sets 12
4.1. Volume-preserving mean curvature flow 13
4.2. Inverse Function Theorem 15
4.3. Another notion of center of mass 18
5. Stability and Foliations 20
5.1. Analyzing the stability operator 20
5.2. Invertibility 21
5.3. Foliations 23
6. Density Theorems 26
6.1. Weighted Sobolev spaces 26
6.2. Scalar curvature equation 28
6.3. Einstein constraint equations 29
6.4. Applications to the center of mass and angular momentum 32
References 33

1. Introduction

Many deep results in mathematical general relativity concern the interplay between globally conserved quantities and the geometric structure of initial data sets, for example: the minimal surface approach by R. Schoen and S.-T. Yau [44, 46] and the spinor method by E. Witten [50] in the proof of the Riemannian positive mass theorem; the inverse mean curvature flow by G. Huisken and T. Ilmanen [34] and the conformal flow by H. Bray [6] in the proof of the Penrose inequality; and the constant mean curvature foliation by G. Huisken and S.-T. Yau [35] (cf. R. Ye [51]) in establishing a geometric notion of center of mass.

In a broad sense, this article is intended to introduce some aspects of the connections between the globally conserved physical quantities, such as the center of mass and angular momentum, and the geometric structure of the manifold, using analysis of the scalar curvature, or more generally the full constraint equations derived from the spacetime Einstein equation. The main focus of the lecture notes is on the constant mean curvature foliations and the geometric center of mass of asymptotically flat initial data sets. This research program was initiated by G. Huisken and S.-T. Yau in 1996 and has drawn great interest in recent years (e.g. [51, 42, 38, 28, 22, 23, 24, 7, 8, 9, 40]). This article begins with a partial survey of the classical results of constant mean curvature surfaces and introduces the now standard concept of stability. We then discuss some recent progress on the constant mean curvature surfaces in asymptotically flat initial data sets and the geometric center of mass. In the last part, we adopt a more analytic approach to study the center of mass and angular momentum from the Einstein constraint equations.

A **spacetime** is an $(n+1)$-dimensional smooth manifold equipped with a Lorentzian metric $\mathbf{g}$ of signature $(- + \cdots +)$. The Einstein equation is the tensor equation

$$\mathrm{Ric}(\mathbf{g}) - \frac{1}{2}R(\mathbf{g})\mathbf{g} = T,$$

where the energy-momentum tensor T physically presents the energy-momentum density of matter. A spacetime is called **vacuum** if it satisfies the Einstein equation with $T = 0$. The prototype vacuum spacetime is Minkowski space $\mathbb{R}^{n+1}$ equipped with the Minkowski metric $\mathbf{g} = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2$. For a general energy-momentum tensor T, we assume the **dominant energy condition**, which is known to hold for physically reasonable matter fields. When expressed in terms of local coordinates, the left hand side of the Einstein equation forms a system of second order equations on the metric components $\mathbf{g}_{\alpha\beta}$. The seminal work of Choquet-Bruhat [25] proved that the left hand side of the Einstein equation can be expressed as a nonlinear hyperbolic operator by using the so-called wave coordinates. Finding a spacetime that satisfies the Einstein equation can then be viewed as the evolution problem for a given initial data set. Thus, it is important to understand the physical and geometric structure of initial data sets.

An **initial data set** for the Einstein equation is a triple (M, g, k), where (M, g) is an n-dimensional Riemannian manifold and k is a symmetric $(0, 2)$ tensor on M. The Gauss-Codazzi equations for submanifolds, along with the Einstein equation, imply that if M is a submanifold in a spacetime with the induced metric g and the induced second fundamental form k, then (M, g, k) must satisfy the following constraint equations

$$\begin{aligned} R(g) - |k|_g^2 + (\mathrm{tr}_g k)^2 &= 2\mu \\ \mathrm{div}_g k - d(\mathrm{tr}_g k) &= J, \end{aligned}$$

where μ is the energy density and J is the momentum density. More specifically, let T be the energy-momentum tensor and let ν be the future-directed timelike normal to M. We define $\mu := T(\nu, \nu)$ and $J := T(\nu, \cdot)$. The dominant energy condition on the tensor T reduces to the inequality $\mu \geq |J|_g$ at each point of M. When $k \equiv 0$, (M, g) is called a **time-symmetric** (or **Riemannian**) initial data set. It is simple to see that in the time-symmetric case the system of constraint equations becomes a single equation $R(g) = 2\mu$. Thus the dominant energy condition coincides with the condition that the scalar curvature of g is nonnegative, which is a condition that naturally appears in Riemannian geometry. Note that, however, for general k the dominant energy condition involves a system of equations and is more complicated.

One family of commonly studied models of isolated gravitational systems is the set of asymptotically flat initial data sets. We say that an initial data set (M, g, k) is **asymptotically flat** (with one end) if there is a compact set $K \subset M$ and a coordinate diffeomorphism

$x : M \setminus K \to \mathbb{R}^n \setminus B$ for some closed ball $B \subset \mathbb{R}^n$ such that, for $i, j = 1, 2, \ldots, n$,

$$g_{ij} - \delta_{ij} = O(|x|^{-q}), \quad k_{ij} = O(|x|^{-1-q}),$$

and such that

$$\mu = O(|x|^{-n-q_0}), \quad J_i = O(|x|^{-n-q_0}),$$

where $q > \frac{n-2}{2}$ and $q_0 > 0$. Here the expression $f(x) = O(|x|^{-q})$ stands for a function satisfying $|f| \leq C|x|^{-q}$ for a constant C depending only on g, k. When a subscript k appears in the expression $f = O_k(|x|^{-q})$, it indicates additional fall-off rates on the derivatives $|\partial^I f(x)| \leq C|x|^{-q-|I|}$ for $|I| = 0, 1, \ldots, k$, but in this article we often omit the subscript k and avoid the discussion about the optimal assumption on regularity.

Note that by definition an asymptotically flat initial data set has trivial topology outside a compact set, but it is shown by J. Isenberg, R. Mazzeo, and D. Pollack [36] that there are no topological obstructions within the compact set.

It is known that asymptotically flat initial data sets possess globally conserved physical quantities. In 1962, R. Arnowitt, S. Deser, and C.W. Misner [1] proposed the definitions of the (total) **energy** E and the **linear momentum** P of an asymptotically flat initial data set (M, g, k) as follows, for $i = 1, 2, \ldots, n$:

$$E = \tfrac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \sum_{j,k=1}^{n} \left(\frac{\partial g_{jk}}{\partial x^k} - \frac{\partial g_{kk}}{\partial x^j} \right) \nu_0^j \, d\mathcal{H}_0^{n-1}$$

$$P_i = \tfrac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \sum_{j=1}^{n} \pi_{ij} \nu_0^j \, d\mathcal{H}_0^{n-1}.$$

Here, the integrals are computed in the coordinate chart $M \setminus K \cong_x \mathbb{R}^n \setminus B$, $\nu_0^j = x^j/|x|$, $\mathcal{H}_0^{n-1}$ is the $(n-1)$-dimensional Euclidean Hausdorff measure, and ω_{n-1} is the volume of the standard unit sphere in $\mathbb{R}^n$. It is known that the scalar E and the vector $(P_1, \cdots, P_n)$ are geometric invariants by R. Bartnik [4] and P. Chruściel [14]. The celebrated positive mass conjecture asserts that the ADM mass is nonnegative $E \geq |P|$. In the time-symmetric case, we may unambiguously use the ADM mass m to denote the energy E, since $|P| = 0$.

There are also the notions of center of mass and angular momentum for an asymptotically flat initial data set. T. Regge and C. Teitelboim [43] and R. Beig and N. Ó Murchadha [5] propose the following definitions of the center of mass $\mathcal{C}_{\text{BORT}}$ and the angular momentum $\mathcal{J}$ (if $E \neq 0$) as follows[1], for $k, \ell = 1, 2, \ldots, n$:

$$\mathcal{C}_\ell = \tfrac{1}{2(n-1)E\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \left[x^\ell \sum_{i,j=1}^{n} \left(\frac{\partial g_{ij}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^j} \right) \nu_0^j - \sum_{i=1}^{n} (g_{i\ell}\nu_0^i - g_{ii}\nu_0^\ell) \right] d\mathcal{H}_0^{n-1} \tag{1.1}$$

$$\mathcal{J}_{(k\ell)} = \tfrac{1}{(n-1)E\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \sum_{i,j=1}^{n} \pi_{ij} Y^i_{(k\ell)} \nu_0^j \, d\mathcal{H}_0^{n-1}, \tag{1.2}$$

where $Y_{(k\ell)} = x^k \frac{\partial}{\partial x^\ell} - x^\ell \frac{\partial}{\partial x^k}$ are the Euclidean rotational vector fields. To distinguish the above definitions from other different notions of center of mass and angular momentum

[1] We remark that in some literature (for example [18]) the BORT center of mass and angular momentum are sometimes defined as $E\mathcal{C}_\ell$ and $E\mathcal{J}_{(k\ell)}$, respectively.

(e.g. [35, 13]), we refer to the integrals (1.1) and (1.2) as the BORT center of mass and the ADM angular momentum, respectively.

In contrast to the ADM energy-momentum, the integrals of $\mathcal{C}_{\text{BORT}}$ and $\mathcal{J}$ are less well-understood and may not even converge in general. In fact, explicit examples of asymptotically flat initial data sets such that the integrals diverge have been constructed [5, 29, 10, 12, 11]. Nevertheless, the author shows that if one assumes the following Regge-Teitelboim conditions, then (1.1) and (1.2) converge and transform correctly with respect to different coordinate charts [27].

An initial data set (M, g, k) is said to satisfy the **Regge-Teitelboim conditions** if it is asymptotically flat and in the coordinate chart $M \setminus K \cong_x \mathbb{R}^n \setminus B$

$$g_{ij}(x) - g_{ij}(-x) = O(|x|^{-1-q}), \quad k_{ij}(x) + k_{ij}(-x) = O(|x|^{-2-q})$$

and

$$\mu(x) - \mu(-x) = O(|x|^{-n-q_0-1}), \quad J_i(x) - J_i(-x) = O(|x|^{-n-q_0-1}).$$

Example 1.1 (Three-dimensional Schwarzschild manifolds)**.** One of the most fundamental examples in general relativity is the Schwarzschild spacetime, which describes the exterior gravitational field of a static, spherically symmetric body. The totally geodesic time-slice outside the apparent horizon of the Schwarzschild spacetime of mass $m > 0$ can be expressed as a Riemannian manifold $M = (2m, \infty) \times \mathbb{S}^2$ endowed with the metric

$$\left(1 - 2ms^{-1}\right)^{-1} ds^2 + s^2 g_{\mathbb{S}^2},$$

where $g_{\mathbb{S}^2}$ is the round metric on the unit sphere. One can readily check that M is the manifold interior of an asymptotically flat initial data set with a minimal boundary and one end, and it has zero scalar curvature. Mathematically one can extend M to a complete asymptotically flat initial data set of zero scalar curvature by "doubling" M across its minimal boundary. The complete two-ended asymptotically flat initial data set can be expressed as a conformally flat metric $(\mathbb{R}^3 \setminus \{a\}, g_{m,a})$, where $g_{m,a} = u^4 g_{\mathbb{E}}$ and

$$u = 1 + \frac{m}{2|x-a|},$$

where $g_{\mathbb{E}}$ is the Euclidean metric. We generally suppress "a" from the notation and write $g_m = g_{m,a}$. One computes directly that m is the ADM energy and a is the BORT center of mass. The asymptotic expansion of g_m for $|x|$ large is

$$g_m = \left(1 + \frac{2m}{|x|} + \frac{2ma \cdot x}{|x|^3} + \frac{3m^2}{2|x|^2} + O(|x|^{-3})\right) g_{\mathbb{E}}.$$

It follows that m appears in the $|x|^{-1}$-term of the expansion and the BORT center of mass appears in the odd part of the $O(|x|^{-2})$-term. This demonstrates that appropriate assumptions need to be imposed on the leading order terms of the data in order for the integrals (1.1) and (1.2) to converge. It explains the motivation behind the definition of the Regge-Teitelboim conditions. We also note that the BORT center of mass of a Schwarzschild manifold is not a point of the manifold. □

This article is organized as follows: In Section 2, we discuss the classical Alexandrov Theorem about embedded constant mean curvature surfaces in Euclidean space. In Section 3, we introduce variational formulas and stability of constant mean curvature surfaces, and then discuss the classical result of Barbosa and do Carmo about uniqueness of stable constant mean curvature surfaces in Euclidean space. In Section 4, we show existence of constant mean curvature surfaces in asymptotically flat initial data sets that are asymptotic to Schwarzschild. We also prove that the geometric center of mass (defined in (4.8)) coincides with the BORT center of mass. In Section 5, we present methods to analyze the spectrum of the stability operator and show that the constant mean curvature surfaces constructed in Section 4 are stable and form a smooth foliation. In Section 6, we discuss density results for the Einstein constraint equations and an application to arbitrarily specifying the BORT center of mass and the ADM angular momentum.

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2. Uniqueness of Embedded CMC Surfaces

A fundamental problem in differential geometry is to characterize the constant mean curvature hypersurfaces in a Riemannian manifold. A classical result due to Alexandrov asserts that the only embedded and closed constant mean curvature surfaces in Euclidean space are the round spheres. The original proof of Alexandrov is based on the arguments which came to be known as the method of moving planes. We instead present another proof due to S. Montiel and A. Ros [39, Section 6.4].

Let (M, g) be an orientable Riemannian manifold, and let $\Sigma^n \subset M^{n+1}$ be an immersed **two-sided** hypersurface, i.e., there exists a globally defined smooth unit normal vector field ν on Σ. The **mean curvature** of Σ with respect to ν is defined by $H := \mathrm{div}_\Sigma \nu$. The mean curvature detects how the (extrinsic) normal vector varies along Σ. According to our convention, the mean curvature of a Euclidean n-sphere is n with respect to the *outward* unit normal vector. An immersed submanifold is said to be **closed** if it is compact and has no boundary and is said to be **embedded** if it has no self-intersection. We first review some basic integral formulas involving mean curvature for closed hypersurfaces.

A **conformal vector field** X is a vector field on M that satisfies

$$L_X g = 2fg,$$

for some function $f : M \to \mathbb{R}$, where L_X is the Lie derivative.

Example 2.1 (cf. [7])**.** Consider the n-dimensional Schwarzschild metric with the ADM mass $m > 0$ of the form $g_m = (1 - 2ms^{2-n})^{-1} ds^2 + s^2 g_{\mathbb{S}^{n-1}}$ on $(s_0, \infty) \times \mathbb{S}^{n-1}$, where $s_0 = (2m)^{\frac{1}{n-2}}$. We change variables and set

$$r(s) = \int_{s_0}^{s} (1 - 2m\tau^{2-n})^{-\frac{1}{2}}\, d\tau.$$

Define $h(r) = s(r)$. We then rewrite the Schwarzschild metric in the form $g_m = dr^2 + h^2(r) g_{\mathbb{S}^{n-1}}$. Define the vector field $X = h(r)\frac{\partial}{\partial r}$. By direct computation,

$$\begin{aligned} L_X g_m &= L_X(dr) \otimes dr + dr \otimes L_X(dr) + X(h^2) g_{\mathbb{S}^{n-1}} \\ &= 2h'(r) dr \otimes dr + 2(h(r))^2 h'(r) g_{\mathbb{S}^{n-1}} \\ &= 2h'(r) g_m. \end{aligned}$$

Thus X is a conformal vector field that satisfies $L_X g_m = 2f g_m$, where

$$f(r) = h'(r) = \left(\frac{dr}{ds}\right)^{-1} = (1 - 2ms^{2-n})^{\frac{1}{2}}.$$

Also note that f satisfies the **static potential equation**

$$(\Delta_{g_m} f) g_m - \text{Hess}(f) + f \text{Ric}_{g_m} = 0.$$

□

In what follows, $d\mu$ will generally denote the induced surface measure on $\Sigma \subset (M, g)$.

Theorem 2.2 (The Generalized Minkowski Integral Formula [7, Proposition 2.3])**.** *Suppose that (M^{n+1}, g) has a conformal vector field X such that $L_X g = 2fg$ for some function f. Let Σ^n be a closed two-sided hypersurface in M, and let H be the mean curvature of Σ with respect to the unit normal vector ν. Then*

$$\int_\Sigma (nf - Hg(X, \nu)) \, d\mu = 0.$$

Proof. Recall that the Lie derivative $L_X g$ in a local frame $\{e_1, \dots, e_{n+1}\}$ has the expression

$$(L_X g)(e_i, e_j) = g(\nabla_{e_i} X, e_j) + g(e_i, \nabla_{e_j} X)$$

for $i, j = 1, 2, \dots, n+1$. We decompose the conformal vector field along Σ into $X = X' + g(X, \nu)\nu$, where $X' \in T\Sigma$. Suppose further that $\{e_1, \dots, e_n\}$ is a local orthonormal frame on Σ, and let ∇ be the covariant derivative of g. Then at each point of Σ

$$\begin{aligned} \text{div}_\Sigma X' + Hg(X, \nu) &= \text{div}_\Sigma X' + g(X, \nu) \sum_{i=1}^n g(\nabla_{e_i} \nu, e_i) \\ &= \sum_{i=1}^n g(\nabla_{e_i} X, e_i) \\ &= \frac{1}{2} \sum_{i=1}^n (L_X g)(e_i, e_i) \\ &= f \sum_{i=1}^n g(e_i, e_i) \\ &= nf. \end{aligned}$$

Integrating the above identity on Σ and applying the divergence theorem yields the desired integral formula. □

Theorem 2.3 (The Heintze-Karcher Inequality, cf. [7, Theorem 3.5])**.** *Let Σ^n be a closed, embedded, two-sided hypersurface in $\mathbb{R}^{n+1}$. Suppose $\Sigma = \partial\Omega$ where Ω is a bounded region in $\mathbb{R}^{n+1}$ with volume $\mathrm{Vol}(\Omega)$, and suppose that the mean curvature H is positive with respect to the outward unit normal. Then*

$$n\int_\Sigma \frac{1}{H}\,d\mu \geq (n+1)\mathrm{Vol}(\Omega)$$

with equality if and only if Σ is a round sphere.

Proof. Consider a deformation $F : \Sigma \times [0,\infty) \to \mathbb{R}^{n+1}$ given by

$$F(x,t) = x - t\nu(x),$$

where ν is the outward unit normal on Σ. Let $\Sigma_t := F(\Sigma, t)$, with surface measure $d\mu$ (suppressing the "t"-dependence). Let $d_\Sigma(p)$ denote the distance of $p \in \mathbb{R}^{n+1}$ to Σ. For t sufficiently small, $\Sigma_t = d_\Sigma^{-1}(t)$ is smooth, but Σ_t may begin to have self-intersection for some t. Hence, instead of working on Σ_t, we consider Σ_t^* defined as follows:

$$\Sigma_t^* = \Sigma_t \cap \{F(x,s) : d_\Sigma(F(x,s+\delta)) = s+\delta \text{ for some } \delta > 0\}.$$

Note that Σ_t^* is a smooth hypersurface contained in Σ_t. Since $\frac{\partial F}{\partial t} = -\nu$. By the variation formulas,

$$\begin{aligned}\frac{\partial}{\partial t}d\mu &= -H\,d\mu\\ \frac{\partial H}{\partial t} &= |A|^2 \geq \frac{H^2}{n}\\ \frac{\partial H^{-1}}{\partial t} &= -H^{-2}|A|^2 \leq -\frac{1}{n}.\end{aligned}$$

Define $Q(t) := n\int_{\Sigma_t^*} H^{-1}\,d\mu$. Then

$$\begin{aligned}Q'(t) &= n\int_{\Sigma_t^*}\left(-H^{-2}|A|^2 + H^{-1}(-H)\right)\,d\mu\\ &\leq n\int_{\Sigma_t^*}(-\frac{1}{n} - 1)\,d\mu\\ &= -(n+1)\int_{\Sigma_t^*}d\mu.\end{aligned}$$

Thus, for $\tau \in (0,\infty)$, we have

$$\begin{aligned}Q(0) - Q(\tau) &= -\int_0^\tau Q'(t)\,dt\\ &\geq (n+1)\int_0^\tau\int_{\Sigma_t^*} d\mu dt\\ &= (n+1)\int_{\{d_\Sigma(x)\leq\tau\}} dx.\end{aligned}$$

Since $Q(\tau) \geq 0$, we obtain the desired inequality by letting $\tau \to \infty$. It is straightforward to verify that Σ is umbilic if the equality holds. ∎

Theorem 2.4 (Alexandrov's Theorem)**.** *Let Σ^n be a closed, embedded, connected, two-sided hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature. Then Σ is a round sphere.*

Remark. Note that the theorem fails if one removes the assumption that Σ is embedded. H. Wente [49] produced an immersed torus of constant mean curvature in $\mathbb{R}^3$. Immersed surfaces of higher genus are constructed by N. Kapouleas [37].

Proof. Note that the position vector field $X = (x^1, \dots, x^{n+1})$ in $\mathbb{R}^{n+1}$ is a conformal vector field (with $f = 1$). By the Minkowski integral formula (Theorem 2.2) and divergence theorem,

$$\begin{aligned}\frac{1}{H}\int_\Sigma n\, d\mu &= \int_\Sigma \langle X, \nu\rangle\, d\mu\\ &= \int_\Omega \mathrm{div}_{\mathbb{R}^{n+1}} X\, dx\\ &= (n+1)\mathrm{Vol}(\Omega).\end{aligned}$$

Thus, we obtain equality in the Heintze-Karcher inequality, which implies that Σ is a round sphere. ☐

Note that the above theorem has been generalized to a large class of warped manifolds, which in particular include the Schwarzschild manifolds by Brendle [7, Theorem 1.1].

3. Stable CMC Surfaces

3.1. **Variational formulas.** Let Σ^n be a smooth closed two-sided hypersurface in (M^{n+1}, g). We are interested in how some geometric quantities on Σ, such as mean curvature, surface area, and enclosed volume, change under deformations of Σ. The relevant formulas are called the variational formulas. Consider a deformation of Σ along its normal direction $F(x,t) : \Sigma \times (-\epsilon, \epsilon) \to M$ satisfying

$$\begin{aligned}\frac{\partial}{\partial t}F(x,t) &= \eta(x,t)\nu(x,t)\\ F(\Sigma, 0) &= \Sigma,\end{aligned}$$

where $\nu(x,t)$ is a unit normal to $\Sigma_t := F(\Sigma, t)$. Define by $F_t(x) = F(x,t)$. We further suppose that $F_t : \Sigma \to M$ is an immersion. By direct computation, the first variation formula says

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{H}^n(\Sigma_t) = \int_\Sigma H\eta\, d\mu, \tag{3.1}$$

where $H = \mathrm{div}_\Sigma \nu$. If one allows arbitrary deformations, then we can strictly decrease the volume of Σ by deforming Σ along $-H\nu$, unless Σ is a minimal hypersurface (i.e. $H \equiv 0$). Thus, minimal hypersurfaces are the critical points of the area functional. The second variation formula at a minimal hypersurface says

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{H}^n(\Sigma_t) = \int_\Sigma \left(|\nabla^\Sigma \eta|^2 - (|A|^2 + \mathrm{Ric}(\nu,\nu))\eta^2\right)\, d\mu, \tag{3.2}$$

where A is the second fundamental form of Σ and Ric is the Ricci tensor of (M, g). Define the stability operator $L_\Sigma := -\Delta_\Sigma - (|A|^2 + \mathrm{Ric}(\nu, \nu))$. A minimal hypersurface is said to be **stable** if $\int_\Sigma \eta L_\Sigma \eta \, d\mu \geq 0$ for all smooth functions η.

If we restrict our attention to a smaller class of the deformations on Σ, hypersurfaces of constant mean curvature $H \neq 0$ can appear as the critical points of the functional $\mathcal{H}^n(\Sigma_t)$. Consider the $(n+1)$-dimensional *signed* volume $V(t)$ between Σ_t and Σ. The volume function satisfies the following variational formula.

Proposition 3.1. *Let Σ^n be a smooth closed two-sided hypersurface in (M^{n+1}, g). Let $F(x,t) : \Sigma \times (-\epsilon, \epsilon) \to M$ satisfy*

$$\frac{\partial}{\partial t} F(x,t) = \eta(x,t)\nu(x,t)$$
$$F(\Sigma, 0) = \Sigma,$$

where $\nu(x,t)$ is a unit normal to $\Sigma_t := F(\Sigma, t)$. Define the volume function by

$$V(t) = \int_{\Sigma \times [0,t]} F^* \, \mathrm{dvol}_M.$$

Then

$$\left.\frac{d}{dt}\right|_{t=0} V(t) = \int_\Sigma \eta \, d\mu.$$

Proof. Let $p \in \Sigma$. Choose a local orthonormal frame $\{e_1, \ldots, e_n, \nu\}$ around $F(p, 0)$. Then $F^* \, \mathrm{dvol}_M = a(t,p) dt \wedge d\mu$, where

$$\begin{aligned} a(t,p) = F^* \, \mathrm{dvol}_M \left(\frac{\partial}{\partial t}, e_1, \ldots, e_n \right) &= \mathrm{dvol}_M \left(\frac{\partial F}{\partial t}, dF_t(e_1), \ldots, dF_t(e_n) \right) \\ = g\left(\frac{\partial F}{\partial t}, \nu(x,t) \right) &= \eta(x,t). \end{aligned}$$

It follows that

$$\left.\frac{d}{dt}\right|_{t=0} V(t) = \int_\Sigma a(0,p) d\mu = \int_\Sigma \eta \, d\mu.$$

□

A variation such that $V(t) = V(0)$ for all $t \in (-\epsilon, \epsilon)$ is called a **volume-preserving variation**. Proposition 3.1 shows that if a variation satisfies $\int_\Sigma \eta \, d\mu = 0$, then it preserves the volume between Σ_t and Σ "infinitesimally" at $t = 0$. In fact, it is shown, on the other hand, that any smooth function $\eta(x)$ on Σ such that $\int_\Sigma \eta \, d\mu = 0$ gives rise to a volume-preserving variation [2, 3].

One can readily see that for volume-preserving variations the hypersurfaces of constant mean curvature are the critical points of the first variational formula. The second variation formula at hypersurfaces of constant mean curvature becomes

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{H}^n(\Sigma_t) = \int_\Sigma \eta L_\Sigma \eta \, d\mu + H \left.\frac{d^2}{dt^2}\right|_{t=0} V(t) = \int_\Sigma \eta L_\Sigma \eta \, d\mu.$$

Therefore, a hypersurface Σ of constant mean curvature is said to be **stable** if and only if

$$\int_\Sigma \eta L_\Sigma \eta \, d\mu \geq 0$$

for all $\eta \in C^\infty(\Sigma)$ such that $\int_\Sigma \eta \, d\mu = 0$. More specifically, define

$$\mu_0 := \inf \left\{ \int_\Sigma \eta L_\Sigma \eta \, d\mu : \eta \in C^\infty(\Sigma), \ \|\eta\|_{L^2(\Sigma)} = 1, \text{ and } \int_\Sigma \eta \, d\mu = 0 \right\}. \tag{3.3}$$

Then Σ is stable if $\mu_0 \geq 0$. From above discussions, stable hypersurfaces are the local minimizers of the area functional $\mathcal{H}^n(\Sigma_t)$ among volume-preserving variations.

Example 3.2. The n-dimensional sphere S_r in $\mathbb{R}^{n+1}$ of radius $r > 0$ centered at the origin is a stable hypersurface of constant mean curvature n/r. The stability operator on S_r is

$$L_0 = -\Delta_0 - (|A|^2 + \mathrm{Ric}(\nu, \nu)) = -\Delta_0 - \frac{n}{r^2},$$

where Δ_0 is the Laplace operator on S_r. Because μ_0 is also an eigenvalue of $-\Delta_0$, by analyzing the eigenvalues of $-\Delta_0$, we obtain $\mu_0 = 0$ with the eigenspace spanned by the coordinate functions $\{x^1, \ldots, x^{n+1}\}$ restricted to S_r. Also note that L_0 is self-adjoint and the cokernel equals its kernel.

Example 3.3. Consider the Schwarzschild manifold $M = (\mathbb{R}^3 \setminus \{\mathcal{C}_{\mathrm{BORT}}\}, g_m)$ where $g_m = u^4 g_{\mathbb{E}}$ and

$$u = 1 + \frac{m}{2|x - \mathcal{C}_{\mathrm{BORT}}|}.$$

For each $r > 0$, let $S_r = \{\mathbb{R}^3 : |x - \mathcal{C}_{\mathrm{BORT}}| = r\}$ be the constant mean curvature sphere homologous to the minimal sphere.

We recall the transformation formula for conformal metrics. Let g_1, g_2 be two metrics on an n-dimensional manifold that are related by $g_2 = u^{\frac{4}{n-2}} g_1$. If ν_1 is a unit normal with respect to g_1, then $\nu_2 = u^{\frac{-2}{n-2}} \nu_1$ is a unit normal with respect to g_2. The corresponding mean curvatures H_1 and H_2 are related by

$$H_2 = u^{\frac{-2}{n-2}} \left(H_1 + \frac{2(n-1)}{n-2} u^{-1} \nabla_{\nu_1} u \right). \tag{3.4}$$

It is not hard to see that umbilicity is preserved under conformal transformation, so the sphere S_r is umbilic in M and has constant mean curvature $\frac{2r-m}{r^2}\phi^{-3}$. This implies that $S_{\frac{m}{2}}$ is a minimal surface and $S_{\frac{(2+\sqrt{3})m}{2}}$ has largest mean curvature. The mean curvature of S_r is increasing in r for $\frac{m}{2} \leq r \leq \frac{(2+\sqrt{3})m}{2}$, and decreasing in r if $r \geq \frac{(2+\sqrt{3})m}{2}$. The stability operator on S_r is given by

$$\begin{aligned} L_{S_r} &= -\Delta_{S_r} - (|A_{S_r}|^2 + \mathrm{Ric}_{g_m}(\nu, \nu)) \\ &= -\phi^{-4}\Delta_0 + \frac{-4r^2 + 8rm - m^2}{2r^4\phi^6}, \end{aligned} \tag{3.5}$$

where Δ_0 is the Laplacian operator of the round sphere of radius r. The smallest eigenvalue of L_{S_r} is

$$\lambda_0 = \frac{-4r^2 + 8rm - m^2}{2r^4\phi^6} = -\frac{2}{r^2} + \frac{10m}{r^3} + O(r^{-4})$$

with the corresponding eigenspace space spanned by constant functions. The next eigenvalues are

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{6m}{r^3\phi^6} = \frac{6m}{r^3} + O(r^{-4})$$

with the corresponding eigenspace spanned by the coordinate functions $\{x^1, x^2, x^3\}$ restricted to S_r. Thus, if $m > 0$, S_r is a stable hypersurface of constant mean curvature (with respect to volume-preserving variations). In fact, the spheres $\{S_r\}$ form a smooth foliation of constant mean curvature spheres with the common center at $\mathcal{C}_{\mathrm{BORT}}$. $\square$

3.2. **Uniqueness of stable CMC surfaces.** A classical result of Barbosa and do Carmo [2] characterizes the stable hypersurfaces in Euclidean space.

Theorem 3.4 (Barbosa-do Carmo [2]). *The only closed, stable, connected, two-sided hypersurfaces in Euclidean space of constant mean curvature are round spheres.*

Proof. Let Σ^n be a hypersurface in $\mathbb{R}^{n+1}$ of constant mean curvature H that satisfies the assumptions in the theorem. Consider the deformation $F : \Sigma \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ given by

$$\frac{\partial F}{\partial t} = \eta\nu,$$

where $\eta = n - H\langle X, \nu\rangle$ and $X = (x^1, \dots, x^{n+1})$ is the position vector of Σ. Then $\int_\Sigma \eta\, d\sigma = 0$ by the Minkowski integral formula (with $f = 1$). Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame along Σ. Note $\nabla_{e_i} X = e_i$ where ∇ is the ambient connection. Hence, at a point where $\nabla^\Sigma_{e_i} e_j = 0$,

$$\begin{aligned}
\Delta_\Sigma \langle X, \nu\rangle &= \sum_{i=1}^n e_i e_i \langle X, \nu\rangle \\
&= \sum_{i=1}^n e_i \left(\langle \nabla_{e_i} X, \nu\rangle + \langle X, \nabla_{e_i}\nu\rangle\right) \\
&= \sum_{i=1}^n \left(\langle \nabla_{e_i} e_i, \nu\rangle + \langle e_i, \nabla_{e_i}\nu\rangle + \langle \nabla_{e_i} X, \nabla_{e_i}\nu\rangle + \langle X, \nabla_{e_i}\nabla_{e_i}\nu\rangle\right) \\
&= \sum_{i=1}^n \left(\langle e_i, \nabla_{e_i}\nu\rangle + \langle X, \nabla_{e_i}\nabla_{e_i}\nu\rangle\right) \\
&= H - |A|^2\langle X, \nu\rangle, \qquad (3.6)
\end{aligned}$$

where in the last equality we use $\sum_i \langle e_k, \nabla_{e_i}\nabla_{e_i}\nu\rangle = 0$ for each k, because H is constant.

Let L_Σ be the stability operator on Σ. The above computation implies that with H constant and $\eta = n - H\langle X, \nu\rangle$,

$$L_\Sigma \eta = -\Delta_\Sigma \eta - |A|^2 \eta = H^2 - n|A|^2.$$

Since Σ is stable, we have since H is constant and $\int_\Sigma \eta\, d\mu = 0$

$$\begin{aligned} 0 \le \int_\Sigma \eta L_\Sigma \eta \, d\mu &= \int_\Sigma (H^2 - n|A|^2)\eta\, d\mu \\ &= -n \int_\Sigma |A|^2 (n - H\langle X, \nu\rangle)\, d\mu \\ &= -n \int_\Sigma (n|A|^2 - H^2)\, d\mu, \end{aligned}$$

where in the last equality we use $\int_\Sigma (H - |A|^2 \langle X, \nu\rangle)\, d\mu = 0$, which is implied by (3.6). Because $n|A|^2 \ge H^2$ with equality if and only if Σ is umbilic, we conclude that Σ is a sphere. □

Remark. Note that the uniqueness result has been generalized to an ambient Riemannian manifold which is complete, simply connected with constant sectional curvature [3]. More precisely, the only stable closed hypersurfaces of constant mean curvature in a complete simply-connected Riemannian manifold with constant sectional curvature are the geodesic spheres.

4. Existence of CMC surfaces in asymptotically flat initial data sets

In 1996, Huisken and Yau initiated a program to study stable constant mean curvature surfaces in asymptotically flat initial data sets. The program is motivated by finding a geometric description of the center of mass in general relativity. We have seen in Example 3.3 the BORT center of mass of Schwarzschild manifold of positive mass is the common geometric center of the (unique) foliation of the stable constant mean curvature surfaces. The goal of Huisken-Yau's program is to find a geometric description of the center of mass for the general case of asymptotically flat initial data sets. Throughout this section, we consider three-dimensional asymptotically flat initial data sets.

We recall that the three-dimensional Schwarzschild metric of mass m is denoted by $g_m = \left(1 + \frac{m}{2|x|}\right)^4 g_{\mathbb{E}}$. Here we are interested in the exterior region of the manifold, so the metric is valid for all $m \in \mathbb{R}$ (not only for $m > 0$). For most of the results presented here, we focus on an asymptotically flat manifold that is close to some Schwarzschild manifold in the following sense.

Definition 4.1. *A three-dimensional asymptotically flat initial data set (M, g) is said to be C^k asymptotic to Schwarzschild of mass m if there is a compact subset $K \subset M$ and a diffeomorphism $M \setminus K \cong_x \mathbb{R}^3 \setminus B$ for a closed ball $B \subset \mathbb{R}^3$ such that*

$$\sum_{|I| \le k} |x|^{2+|I|} \left| \partial^I \left(g_{ij}(x) - (g_m)_{ij}(x) \right) \right| \le C$$

for some constant $C > 0$.

Remark. The assumptions on k for regularity C^k vary among different results that we discuss below, but we omit the precise assumptions on C^k in their statements.

Theorem 4.2 (Huisken-Yau [35])**.** *Let (M, g) be asymptotic to Schwarzschild of mass $m > 0$. Then there exists a foliation of stable constant mean curvature surfaces in the exterior region of M.*

The proof of Huisken and Yau consists of two parts. For the existence part, they use the volume-preserving mean curvature flow to evolve a sufficiently round initial surface into a constant mean curvature surface. Next, using the estimates obtained from the flow, they analyze the eigenvalues of the stability operator and show that the constant mean curvature surfaces are stable and form a smooth foliation. We sketch the method of volume-preserving mean curvature flow in Section 4.1 and discuss the eigenvalue estimates in Section 5. Note that a different approach by Ye [51] uses the inverse function theorem for the existence part, which we discuss in Section 4.2

4.1. **Volume-preserving mean curvature flow.** The volume-preserving mean curvature flow is a normalized mean curvature flow. It was first introduced by Huisken in the Euclidean setting [33]. The flow is designed specifically to keep the enclosed volume the same and to decrease the surface area under the flow.

Let (M, g) be asymptotic to Schwarzschild. Denote by $S_r = \{x : |x| = r\}$ the coordinate sphere in M. For each r sufficiently large, we define the volume-preserving mean curvature flow $F_r : \mathbb{S}^2 \times [0, T) \to M$ as follows, for $t \geq 0, p \in \mathbb{S}^2$:

$$\begin{aligned} \frac{\partial}{\partial t} F_r(p,t) &= (\overline{H} - H)\nu(p,t) \\ F_r(\mathbb{S}^2, 0) &= S_r, \end{aligned} \tag{4.1}$$

where $\overline{H} = |\Sigma_t|^{-1} \int_{\Sigma_t} H \, d\mu$, $\Sigma_t = F_r(\mathbb{S}^2, t)$, and $|\Sigma_t|$ is the area of Σ_t. By Proposition 3.1, the flow keeps the signed volume between Σ_t and S_r the same. Furthermore, the first variation formula implies

$$\frac{d}{dt}|\Sigma_t| = -\int_{\Sigma_t} (H - \overline{H})^2 \, d\mu.$$

It implies that the area of Σ_t is strictly decreasing unless H is a constant. Therefore, if the flow exists for all time, Σ_t converges to a constant mean curvature surface.

Note that the volume-preserving mean curvature flow (4.1) is a quasi-linear parabolic system, so it has a unique short-time solution for a smooth initial surface. However, the flow may develop singularities at a finite time. For surfaces in Euclidean space that are uniformly convex, it is shown that the flow exists for all time and converges to a round sphere [33]. For surfaces in an initial data set that is asymptotic to Schwarzschild, we have the following result.

Theorem 4.3 (Huisken-Yau [35])**.** *Let (M, g) be asymptotic to Schwarzschild of $m > 0$. There exist positive constants r_0, C depending only on g, such that for all $r \geq r_0$, the volume-preserving mean curvature flow* (4.1) *has a unique smooth solution for all time. Furthermore, Σ_t converge exponentially fast to an embedded surface Σ of constant mean curvature H, and for $x \in \Sigma$,*

$$\begin{aligned} &||x| - r| \leq C \\ &\left| H - \frac{2}{r} + \frac{4m}{r^2} \right| \leq C r^{-2}. \end{aligned}$$

The main ingredient of the proof is to show that the solution Σ_t stays in a class of sufficiently round surfaces, and hence it does not develop singularities along the flow.

For a surface Σ in (M,g), denote by $\mathring{A} := A - \frac{1}{2}Hg_\Sigma$ the traceless part of the second fundamental form A of Σ, where g_Σ is the induced metric. Let r be large and $B_0, B_1, B_1 > 0$. Define the class $\mathfrak{B}_r(B_0, B_1, B_2)$ of smooth closed surfaces of genus zero in (M,g) by

$$\mathfrak{B}_r(B_0, B_1, B_2) = \{\Sigma \subset M: \text{ for all } x \in \Sigma,\ ||x| - r| \leq B_0,\ |\mathring{A}| \leq B_1 r^{-3},\ |\nabla^\Sigma \mathring{A}| \leq B_2 r^{-4}\}.$$

By carefully choosing the constants B_0, B_1, B_2 (depending on the *a priori* estimates of $|\mathring{A}|$ and $|\nabla \mathring{A}|$ along the flow), there exists r_0 sufficiently large (depending on B_0, B_1, B_2 and g) such that for each $r \geq r_0$, the solution Σ_t to (4.1) remains in $\mathfrak{B}_r(B_1, B_2, B_3)$. This then implies long-time existence of the solutions. The proof is beyond the scope of this article, so we refer the readers to the original paper [35, Section 3].

Here we explain the motivation behind the smallness assumptions on $|\mathring{A}|$ and $|\nabla \mathring{A}|$ in the definition of $\mathfrak{B}_r(B_0, B_1, B_2)$. Note that if $|\mathring{A}| = 0$, then all the principal curvatures at each point of the hypersurface are equal and hence the hypersurface is umbilic. It is known that the only closed umbilic hypersurfaces in Euclidean space are round spheres. (We applied this fact earlier in the proof of Theorem 3.4.) For surfaces that are *almost* umbilic, there are several quantitative versions that measure how far the surfaces are from being round (see, for example, [20]). Below we provide a simple version of the quantitative estimates.

We define the **area radius** for a closed surface Σ in (M,g) by

$$r_\Sigma := \sqrt{\frac{|\Sigma|}{4\pi}},$$

Proposition 4.4 (cf. [35, Proposition 2.1])**.** *Let Σ be a closed surface in $\mathbb{R}^3$ of genus zero. Let $B_1, B_2 > 0$ be real numbers. Suppose*

$$|\mathring{A}| \leq B_1 r_\Sigma^{-3}, \quad |\nabla \mathring{A}| \leq B_2 r_\Sigma^{-4}.$$

There exists an absolute constant $C > 0$ such that if $r_\Sigma > C(\sqrt{B_1} + \sqrt{B_2})$, then the principal curvatures λ_1, λ_2 satisfy, for $i = 1, 2$,

$$\left|\lambda_i - \frac{\overline{H}}{2}\right| \leq C(B_1 + B_2) r_\Sigma^{-3}.$$

Proof. In the proof, C is assumed to be an absolute constant and may change from line to line. As a consequence of the Codazzi equation [32, Lemma 2.2], we have

$$|\nabla A|^2 \geq \frac{3}{4}|\nabla H|^2.$$

Together with the assumption on $|\nabla \mathring{A}|$, this implies an upper bound on $|\nabla H|$:

$$|\nabla \mathring{A}|^2 = |\nabla A|^2 - \frac{1}{2}|\nabla H|^2 \geq \frac{1}{4}|\nabla H|^2.$$

Let $x_0 \in \Sigma$ such that $H(x_0) = \overline{H}$. By the mean value theorem and the above bound on $|\nabla H|$, we have

$$|H(x) - \overline{H}| \leq \sup_{x \in \Sigma} |\nabla H(x)| d \leq 2B_2 r_\Sigma^{-4} d, \tag{4.2}$$

where d is the *intrinsic* diameter of Σ. Note that d can be estimated in terms of the mean curvature [48, Theorem 1.1] as follows:

$$d \le C \int_\Sigma |H|\, d\mu \le C(|\overline{H}| r_\Sigma^2 + B_2 r_\Sigma^{-2} d),$$

where we apply (4.2) in the last inequality. Choosing $r_\Sigma \ge \sqrt{2CB_2}$ yields that $d \le 2C|\overline{H}|r_\Sigma^2$. Using (4.2) again, we have for all $x \in \Sigma$

$$|H(x) - \overline{H}| \le CB_2 r_\Sigma^{-2} |\overline{H}|.$$

By the Gauss-Bonnet theorem and the assumption on $|\mathring{A}|$, we have

$$\begin{aligned} |\overline{H}| &\le |\Sigma|^{-\frac{1}{2}} \left(\int_\Sigma H^2 \, d\mu \right)^{\frac{1}{2}} \\ &\le |\Sigma|^{-\frac{1}{2}} \left(\int_\Sigma 2|\mathring{A}|^2 \, d\mu + 4 \int_\Sigma K \, d\mu \right)^{\frac{1}{2}} \\ &\le C r_\Sigma^{-1}, \end{aligned}$$

provided $r_\Sigma \ge \sqrt{B_1}$. Thus we obtain

$$|H(x) - \overline{H}| \le CB_2 r_\Sigma^{-3}.$$

By the assumption that $|\mathring{A}| \le B_1 r_\Sigma^{-3}$, we conclude for $i = 1, 2$,

$$\left| \lambda_i - \frac{\overline{H}}{2} \right| \le C(B_1 + B_2) r_\Sigma^{-3}.$$

□

4.2. **Inverse Function Theorem.** An alternative method to construct a constant mean curvature surface is by graphically perturbing an initial surface whose mean curvature is almost constant.

Let (M, g) be asymptotic to Schwarzschild of mass m. Let

$$p_{ij}(x) := g_{ij}(x) - \left(1 + \frac{2m}{|x|}\right) \delta_{ij}.$$

For r sufficiently large, let $S_r(a)$ be a coordinate sphere defined by $S_r(a) = \{x \in M : |x - a| = r\}$. Denote $\rho^i = \frac{x^i - a^i}{r}$. By direct computation [27, (5.1)], the mean curvature of $S_r(a)$ at x in $S_r(a)$ is

$$\begin{aligned} H_{S_r(a)} &= \frac{2}{r} - \frac{4m}{r^2} + \frac{6m(x-a)\cdot a}{r^4} + \frac{9m^2}{r^3} + \frac{1}{2} p_{ij,k}(x) \rho^i \rho^j \rho^k + 2 \frac{p_{ij}(x)}{r} \rho^i \rho^j \\ &\quad - p_{ij,i}(x) \rho^j - \frac{p_{ii}(x)}{r} + \frac{1}{2} p_{ii,j}(x) \rho^j + O(r^{-4}(1 + |a|)) \\ &=: \frac{2}{r} - \frac{4m}{r^2} + \frac{6m(x-a)\cdot a}{r^4} + \frac{9m^2}{r^3} + G_r(x, a). \end{aligned} \tag{4.3}$$

Throughout this section, we use the Einstein summation convention and sum over repeated indices, and a comma denotes a partial derivative. From (4.3), the mean curvature of the coordinate sphere is almost constant, up to terms of order $O(r^{-3})$. We show below that if

$m \neq 0$, one can find a surface of constant mean curvature near $S_r(a)$ for a suitable chosen vector a (depending on r). Furthermore, the center a converges to the BORT center of mass as r tending to infinity.

Theorem 4.5 (Ye [51], Huang [27])**.** *Let (M, g) be asymptotic to Schwarzschild of mass $m \neq 0$. There exist positive constants r_0 and C, depending only on g, such that for each $r \geq r_0$, there exists a surface Σ_r of constant mean curvature $\frac{2}{r} - \frac{4m}{r^2}$, and Σ_r can be expressed as a normal graph over the coordinate sphere*

$$\Sigma_r = \{x + \phi(x)\nu_{S_r}(x) : x \in S_r(\mathcal{C}_{\mathrm{BORT}})\}$$

for some $\phi \in C^{2,\alpha}(S_r(\mathcal{C}_{\mathrm{BORT}}))$ which satisfies $\sum_{|I|\leq 2} r^{|I|}|\partial^I \phi| + \sum_{|I|=2} r^{2+\alpha}[\partial^I \phi]_\alpha \leq Cr^{-1}$, where ν_{S_r} is the outward unit normal vector on S_r with respect to g.

Sketch of Proof. In the following we suppress the subscript r in Σ_r when the context is clear. Fix an asymptotically flat coordinate system in the exterior region of M. Let Σ be a graph over the coordinate sphere: for $\phi \in C^{2,\alpha}(S_r(a))$ suitably small,

$$\Sigma = \{x + \phi\nu_{S_r} : x \in S_r(a)\}.$$

Fix r sufficiently large, which will be specified later. Denote by $\mathcal{H}_r(a, \phi) : \mathbb{R}^3 \times C^{2,\alpha}(S_r(a)) \to C^{0,\alpha}(S_r(a))$ the mean curvature operator that sends the function ϕ to the mean curvature of the normal graph Σ in (M, g). By Taylor expansion in the ϕ-component,

$$\mathcal{H}_r(a, \phi) = \mathcal{H}_r(a, 0) + d\mathcal{H}_r(a, 0)(\phi) + \int_0^1 \left(d\mathcal{H}_r(a, s\phi) - d\mathcal{H}_r(a, 0)\right)(\phi)\, ds, \tag{4.4}$$

where $d\mathcal{H}_r$ is the first Fréchet derivative with respect to the second component. Specifically, $d\mathcal{H}_r(a, 0)$ is the stability operator on $S_r(a)$ with respect to g:

$$d\mathcal{H}_r(a, 0) = -\Delta_{S_r(a)} - (|A|^2 + \mathrm{Ric}(\nu_{S_r}, \nu_{S_r})) =: L_{S_r(a)}.$$

Observe that the term $\mathcal{H}_r(a, 0)$ in (4.4) is the mean curvature of coordinate sphere computed as in (4.3). Thus, solving $\mathcal{H}_r(a, \phi) = \frac{2}{r} - \frac{4m}{r^2}$ for some (a, ϕ) is equivalent to solving

$$L_{S_r(a)}\phi = -\frac{6m(x-a)\cdot a}{r^4} - \frac{9m^2}{r^3} - G_r(x, a) - \int_0^1 \left(d\mathcal{H}_r(a, s\phi) - d\mathcal{H}_r(a, 0)\right)(\phi)\, ds. \tag{4.5}$$

Since (M, g) is asymptotic to Schwarzschild, on the coordinate sphere $S_r(a)$ we have

$$|A|^2 = \frac{2}{r^2} + O(r^{-3}), \quad \mathrm{Ric}(\nu_{S_r}, \nu_{S_r}) = O(r^{-3}),$$

provided that $|a|$ is bounded by a constant independent of r. Hence we replace the stability operator $L_{S_r(a)}$ by the stability operator $L_0 := -\Delta_0 - \frac{2}{r^2}$ on the Euclidean round sphere, where Δ_0 is the Laplace operator on the Euclidean round sphere $S_r(a)$. We then rewrite (4.5) as a differential equation on the Euclidean round sphere:

$$\begin{aligned} L_0\phi &= -\frac{6m(x-a)\cdot a}{r^4} - \frac{9m^2}{r^3} - G_r(x, a) + O(r^{-1}|\partial^2\phi| + r^{-2}|\partial\phi| + r^{-3}|\phi|) \\ &=: F_r(x, a, \phi, \partial\phi, \partial^2\phi). \end{aligned} \tag{4.6}$$

A *necessary* condition for the above equation to have a solution is that F_r must be perpendicular to the cokernel of L_0, which is spanned by $\{x^1 - a^1, x^2 - a^2, x^3 - a^3\}$ restricted on $S_r(a)$ (see Example 3.2). By the following lemma, the parameter a can be chosen to accomplish this.

Lemma 4.6 (Huang [27, Lemma 5.1])**.** *Let (M, g) be asymptotic to Schwarzschild of mass m. There exists r_0 sufficiently large such that for each $r \geq r_0$ and for for each $i = 1, 2, 3$*

$$\int_{S_r(a)} (x^i - a^i) G_r(x, a)\, d\mu_0 = -8\pi m\, \mathcal{C}^i_{\mathrm{BORT}} + O(r^{-1}),$$

where $G_r(x, a)$ is the remainder term in (4.3) and $d\mu_0$ is the area measure of the Euclidean round sphere.

Remark. The above lemma has been generalized to initial data sets with the Regge-Teitelboim conditions. We prove it in Lemma 4.9 below.

By Lemma 4.6 and direct computation, we obtain

$$\int_{S_r(a)} F_r(x, a, \phi, \partial\phi, \partial^2\phi)(x^i - a^i)\, d\mu_0 = -8\pi m(a^i - \mathcal{C}^i_{\mathrm{BORT}}) + O(r^{-1}\|\phi\|_{C^2}), \quad i = 1, 2, 3.$$

If $m \neq 0$, we choose $a = \mathcal{C}_{\mathrm{BORT}} + O(r^{-1}\|\phi\|_{C^2})$ such that the above integral vanishes. Thus, $F_r(x, a, \phi, \partial\phi, \partial^2\phi)$ belongs to the range of L_0.

Next we use the Schauder Fixed Point Theorem to find a solution to (4.6).

Theorem 4.7 (Schauder Fixed Point Theorem, e.g. [26, Chapter 11])**.** *Let $\mathcal{B}$ be a compact convex subset in a Banach space, and let $T : \mathcal{B} \to \mathcal{B}$ be a continuous map. Then T has a fixed point, that is, $Tx = x$ for some $x \in \mathcal{B}$.*

Define the convex subset $\mathcal{B} \subset C^2(S_r(a))$ by $\mathcal{B} := \{u \in C^2(S_r(a)) : \|u\|_{C^{2,\alpha}} \leq 1\}$. Note that $\mathcal{B}$ is compact by the Arzela-Ascoli Theorem. Given $w \in C^2(S_r(a))$, we have shown that there exists a vector a such that $F_r(x, a, w, \partial w, \partial^2 w)$ belongs to the range of L_0. It implies that there exists a solution $v \in C^{2,\alpha}(S_r(a))$ such that

$$L_0 v = F_r(x, a, w, \partial w, \partial^2 w). \tag{4.7}$$

Define the map $T : \mathcal{B} \to C^2(S_r(a))$ by $T(w) = v$, where v is the unique solution to (4.7) such that v is perpendicular to the kernel of L_0. One can verify that T is continuous. By the Schauder estimates for solutions perpendicular to the kernel, we obtain

$$\|v\|_{C^{2,\alpha}(S_r(a))} \leq C\|F_r(x, a, w, \partial w, \partial^2 w)\|_{C^{0,\alpha}(S_r(a))} \leq Cr^{-1}\|w\|_{C^{2,\alpha}(S_r(a))},$$

where the constant C depends only on the metric g. Choose r such that $r \geq C$. It follows that T maps $\mathcal{B}$ into itself. Thus, by Schauder Fixed Point Theorem, T has a fixed point ϕ. Then ϕ solves the desired equation

$$\mathcal{H}_r(a, \phi) = \frac{2}{r} - \frac{4m}{r^2},$$

where $a = \mathcal{C}_{\mathrm{BORT}} + O(r^{-1}\|\phi\|_{C^2})$. □

Let $\{\Sigma_r\}$ be the family of constant mean curvature surfaces constructed in the previous theorem, and let $\{x^1, x^2, x^3\}$ be the coordinate functions. The geometric center of mass of (M, g) proposed by Huisken-Yau is defined as follows, for $i = 1, 2, 3$:

$$\mathcal{C}^i_{\mathrm{Geom}} = \lim_{r\to\infty} \frac{\int_{\Sigma_r} x^i\, d\mu_0}{\int_{\Sigma_r} d\mu_0}, \tag{4.8}$$

where $d\mu_0$ is the Euclidean area measure.

Corollary 4.8 (Huang [27, Theorem 2]). *Let (M, g) be asymptotic to Schwarzschild of $m \neq 0$. Then these notions of the center of mass coincide*

$$\mathcal{C}_{\text{BORT}} = \mathcal{C}_{\text{Geom}}.$$

We remark that the above corollary has been generalized to the class of asymptotically flat initial data sets that satisfy the Regge-Teitelboim conditions [28].

4.3. **Another notion of center of mass.** In this section we show the following identity between the mean curvature of the coordinate spheres and the BORT center of mass.

Lemma 4.9 (Huang [28]). *Let (M, g, k) be an asymptotically flat initial data set satisfying the Regge-Teitelboim conditions. Given $a \in \mathbb{R}^3$, denote by $S_r(a) = \{x \in M : |x - a| = r\}$ the coordinate sphere centered at a of radius r. Then,*

$$\int_{S_r(a)} (x^\alpha - a^\alpha)\left(H - \frac{2}{r}\right) d\mu_0 = 8\pi E(a^\alpha - \mathcal{C}^\alpha_{\text{BORT}}) + O(r^{1-2q}), \quad \alpha = 1, 2, 3, \tag{4.9}$$

where H is the mean curvature of $S_r(a)$ with respect to g and $d\mu_0$ is the area measure of the Euclidean round sphere $S_r(a)$.

Proof. Denote by $h_{ij} = g_{ij} - \delta_{ij}$ and $\rho^i = \frac{x^i - a^i}{r}$. Throughout the proof, we use the Einstein summation convention and sum over repeated indices. By direct computation [28, Lemma 2.1], we find for $x \in S_r(a)$,

$$H(x) = \frac{2}{r} + \frac{1}{2} h_{ij,k}(x)\rho^i\rho^j\rho^k + 2h_{ij}(x)\frac{\rho^i\rho^j}{r} - h_{ij,i}(x)\rho^j + \frac{1}{2}h_{ii,j}(x)\rho^j - \frac{h_{ii}(x)}{r} + E_0(x),$$

where $E_0(x) = O(r^{-1-2q})$ and $E_0(x) - E_0(-x) = O(r^{-2-2q})$. We state the following key identity

$$\begin{aligned} &\int_{S_r(a)} (x^\alpha - a^\alpha)\frac{1}{2}h_{ij,k}(x)\rho^i\rho^j\rho^k \, d\mu_0 \\ &= \int_{S_r(a)} (x^\alpha - a^\alpha)\left(\frac{1}{2}h_{ij,i}(x)\rho^j - 2h_{ij}(x)\frac{\rho^i\rho^j}{r}\right) d\mu_0 + \int_{S_r(a)} \frac{1}{2}\left(h_{ii}(x)\rho^\alpha + h_{i\alpha}(x)\rho^i\right) d\mu_0. \end{aligned} \tag{4.10}$$

Assuming the above identity, we obtain

$$
\begin{aligned}
&\int_{S_r(a)} (x^\alpha - a^\alpha)\left(H(x) - \frac{2}{r}\right) d\mu_0 \\
&= -\frac{1}{2}\int_{S_r(a)} (x^\alpha - a^\alpha)(h_{ij,i} - h_{ii,j})\rho^j \, d\mu_0 + \frac{1}{2}\int_{S_r(a)} (h_{i\alpha}\rho^i - h_{ii}\rho^\alpha)\, d\mu_0 + O(r^{1-2q}) \\
&= -\frac{1}{2}\int_{S_r(a)} \left[x^\alpha(g_{ij,i} - g_{ii,j})\rho^j - (g_{i\alpha}\rho^i - g_{ii}\rho^\alpha)\right] d\mu_0 \\
&\quad + \frac{1}{2}a^\alpha \int_{S_r(a)} (g_{ij,i} - g_{ii,j})\rho^j \, d\mu_0 + O(r^{1-2q}) \\
&= -\frac{1}{2}\int_{S_r(a)} \left[x^\alpha(g_{ij,i} - g_{ii,j})\frac{x^j}{r} - \left(g_{i\alpha}\frac{x^i}{r} - g_{ii}\frac{x^\alpha}{r}\right)\right] d\mu_0 \\
&\quad + \frac{1}{2}a^\alpha \int_{S_r(a)} (g_{ij,i} - g_{ii,j})\frac{x^j}{r}\, d\mu_0 + O(r^{1-2q}),
\end{aligned}
$$

where we use the Regge-Teitelboim conditions in all the equalities. The desired identity follows from the definitions of the ADM energy and the BORT center of mass.

It remains to prove (4.10). Our original proof uses a density theorem (Theorem 6.2) which states that initial data sets with harmonic asymptotics are dense among initial data sets with the Regge-Teitelboim conditions in a suitable topology such that the ADM energy and the BORT center of mass vary continuously. It is then straightforward to verify that (4.10) holds for initial data sets with harmonic asymptotics. Eichmair and Metzger later gave the following proof [24]. For each α, denote the vector field $X_{(\alpha)} = (x^\alpha - a^\alpha)h_{ij}\rho^i\partial_j$. By the first variation formula,

$$
\int_{S_r(a)} \operatorname{div}_0 X_{(\alpha)}\, d\mu_0 = \int_{S_r(a)} H_0\langle X_{(\alpha)}, \rho\rangle\, d\mu_0 = \int_{S_r(a)} 2(x^\alpha - a^\alpha)\frac{h_{ij}\rho^i\rho^j}{r}\, d\mu_0,
$$

where div_0 is the divergence operator on the Euclidean round sphere $S_r(a)$. Then (4.10) follows from the direct computation:

$$
\begin{aligned}
\operatorname{div}_0 X_{(\alpha)} &= \left(\delta_{ij} - \rho^i\rho^j\right)\partial_i X^j_{(\alpha)} \\
&= h_{i\alpha}\rho^i + (x^\alpha - a^\alpha)\left(\frac{h_{ii}}{r} - 2\frac{h_{ij}}{r}\rho^i\rho^j + h_{ij,j}\rho^i - h_{ij,k}\rho^i\rho^j\rho^k\right).
\end{aligned}
$$

□

The above lemma gives us a new notion of center of mass that involves the mean curvature: for $i = 1, 2, 3$,

$$
\mathcal{C}_H^\alpha = \lim_{r\to\infty} -\frac{1}{8\pi E}\int_{S_r(0)} x^\alpha H\, d\mu_0,
$$

where H is the mean curvature of the coordinate sphere $S_r(0)$ with respect to g, and $d\mu_0$ is the area measure of a Euclidean round sphere.

Corollary 4.10. *Let (M, g, k) be an asymptotically flat initial data set satisfying the Regge-Teitelboim conditions. Then these notions of center of mass coincide*

$$
\mathcal{C}_H = \mathcal{C}_{\text{BORT}}.
$$

5. Stability and Foliations

After having obtained a family of constant mean curvature surfaces in Section 4, we now discuss their properties in this section. We continue to restrict our discussions to three-dimensional asymptotically flat initial data sets throughout this section, unless otherwise specified.

5.1. **Analyzing the stability operator.**

We have shown in Section 4 the existence of a family of constant mean curvature surfaces in an initial data set (M, g) asymptotic to Schwarzschild. From the construction, each member of the family of constant mean curvature surfaces $\{\Sigma_r\}$ can be expressed as a normal graph over the corresponding coordinate sphere at a common center a as below:

$$\Sigma_r = \{x + \phi\nu_{S_r} : x \in S_r(a)\}, \tag{$\star$}$$

where $\phi \in C^{2,\alpha}(S_r(a))$ depends on r and $\sum_{|I|\leq 2} r^{|I|}|\partial^I\phi| + \sum_{|I|=2} r^{2+\alpha}[\partial^I\phi]_\alpha \leq Cr^{-1}$. In particular, each Σ_r satisfies the following properties:

$$\begin{aligned} H &= \frac{2}{r} - \frac{4m}{r^2} \\ |A|^2 + \mathrm{Ric}(\nu,\nu) &= \frac{2}{r^2} - \frac{10m}{r^3} + O(r^{-4}) \\ |\Sigma_r| &= 4\pi r^2 + O(r) \\ K &\geq \frac{1}{r^2} - \frac{2m}{r^3} - Cr^{-4}, \end{aligned} \tag{$\star\star$}$$

where K is the Gauss curvature of Σ_r and ν is the unit normal to Σ_r, and where $f = O(r^q)$ denotes a function satisfying $|f| \leq Cr^q$ for all r, where $C > 0$ depends only on g.

Below we show that for a family of surfaces satisfying the properties $(\star\star)$, there exists r_0 sufficiently large such that for each $r \geq r_0$, the constant mean curvature surface Σ_r is stable. We first recall a classical estimate on the first nonzero eigenvalue of the Laplace operator.

Lemma 5.1 (Lichnerowicz)**.** *Let Σ be an n-dimensional closed Riemannian manifold. Let λ_{Lap} be the first nonzero eigenvalue of the Laplace operator $-\Delta$. Suppose that for some constant $\kappa > 0$ the Ricci curvature satisfies*

$$\mathrm{Ric}(\xi,\xi) \geq (n-1)\kappa|\xi|^2,$$

for all $\xi \in TM$. Then $\lambda_{\mathrm{Lap}} \geq n\kappa$.

Remark. The above equality holds if and only if Σ is isometric to the n-sphere of constant sectional curvature κ [41].

Proof. Recall the Bochner-Lichnerowicz identity:

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \langle\nabla u, \nabla\Delta u\rangle + \mathrm{Ric}(\nabla u, \nabla u).$$

Let u be an eigenfunction corresponding to λ_{Lap} such that $-\Delta u = \lambda_{\mathrm{Lap}} u$. Integrating the identities over Σ and using $|\mathrm{Hess}(u)|^2 \geq (\Delta u)^2/n$, we obtain $\int_\Sigma |\nabla u|^2\, d\mu = \lambda_{\mathrm{Lap}} \int_\Sigma u^2\, d\mu$ as well as

$$\begin{aligned}
0 &= \frac{1}{2}\int_\Sigma \Delta|\nabla u|^2\, d\mu \\
&\geq \int_\Sigma \left[\frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla \Delta u\rangle + \mathrm{Ric}(\nabla u, \nabla u)\right] d\mu \\
&\geq \left(\frac{\lambda_{\mathrm{Lap}}^2}{n} + ((n-1)\kappa - \lambda_{\mathrm{Lap}})\lambda_{\mathrm{Lap}}\right)\int_\Sigma u^2\, d\mu.
\end{aligned}$$

Since $\lambda_{\mathrm{Lap}} > 0$, the desired inequality follows.

□

Theorem 5.2 (Huisken-Yau [35]). *Let (M, g) be asymptotic to Schwarzschild of mass m. Suppose $\{\Sigma_r\}$ is a family of surfaces satisfying the properties $(\star\star)$. Then there exists $C > 0$ (depending only on g) such that for each Σ_r*

$$\mu_0 \geq \frac{6m}{r^3} - Cr^{-4},$$

where μ_0 is defined by (3.3). As a consequence, if $m > 0$, there exists r_0 sufficiently large such that for each $r \geq r_0$, we have $\mu_0 > 0$ and hence Σ_r is stable.

Proof. Applying Lemma 5.1 to a two-dimensional surface Σ yields that $\lambda_{\mathrm{Lap}} \geq 2\kappa$ where κ is the minimum of the Gauss curvature of Σ. Then on Σ_r we have by $(\star\star)$

$$\lambda_{\mathrm{Lap}} \geq \frac{2}{r^2} - \frac{4m}{r^3} - Cr^{-4}.$$

The proposition follows from the definition of μ_0 and the properties $(\star\star)$. □

5.2. **Invertibility.** To show that the stability operator is invertible, we analyze the eigenvalues of the operator.

Theorem 5.3 (cf. Huisken-Yau [35, Theorem 4.1]). *Let (M, g) be asymptotic to Schwarzschild of mass m. Suppose $\{\Sigma_r\}$ is a family of surfaces satisfying the properties $(\star\star)$. Let λ_0 be the lowest eigenvalue of the stability operator L_{Σ_r} on Σ_r, and let λ_1 be the next eigenvalue. Then there exists $C > 0$ (depending only on g) such that*

$$\begin{aligned}
\lambda_0 &= -\frac{2}{r^2} + \frac{10m}{r^3} + O(r^{-4}), \\
\lambda_1 &\geq \frac{6m}{r^3} - Cr^{-4}.
\end{aligned}$$

As a consequence, if $m > 0$, there exists r_0 sufficiently large such that for each $r \geq r_0$, the stability operator $L_{\Sigma_r} : C^{2,\alpha}(\Sigma_r) \to C^{0,\alpha}(\Sigma_r)$ is a linear isomorphism.

Proof. Let w be an eigenfunction for λ_0 so that

$$L_{\Sigma_r} w = -\Delta w - (|A|^2 + \mathrm{Ric}(\nu, \nu))w = \lambda_0 w. \tag{5.1}$$

Multiplying the above identity by w and integrating it over Σ_r, we have

$$\begin{aligned}\lambda_0 \int_{\Sigma_r} w^2\, d\mu &= \int_{\Sigma_r} \left[|\nabla w|^2 - (|A|^2 + \mathrm{Ric}(\nu,\nu)) w^2 \right] d\mu \\ &\geq \left(-\frac{2}{r^2} + \frac{10m}{r^3} - Cr^{-4} \right) \int_{\Sigma_r} w^2\, d\mu,\end{aligned}$$

where we use $|\nabla w| \geq 0$ and apply the properties ($\star\star$) to the term involving $(|A|^2 + \mathrm{Ric}(\nu,\nu))$. On the other hand, using the constant function in the Rayleigh quotient yields

$$\lambda_0 \leq -\frac{2}{r^2} + \frac{10m}{r^3} + Cr^{-4}.$$

Thus we have shown that

$$\lambda_0 = -\frac{2}{r^2} + \frac{10m}{r^3} + O(r^{-4}). \tag{5.2}$$

Moreover, the function w is almost constant in the L^2 sense. Let $\overline{w} = |\Sigma_r|^{-1} \int_{\Sigma_r} w\, d\mu$ denote the mean value of w. Multiply (5.1) by $(w - \overline{w})$ and integrate to obtain

$$\begin{aligned}\int_{\Sigma_r} |\nabla (w - \overline{w})|^2\, d\mu &= \int_{\Sigma_r} (\lambda_0 + |A|^2 + \mathrm{Ric}(\nu,\nu))(w - \overline{w})^2\, d\mu \\ &\quad + \int_{\Sigma_r} (|A|^2 + \mathrm{Ric}(\nu,\nu)) \overline{w} (w - \overline{w})\, d\mu.\end{aligned}$$

Using the estimates of $\lambda_{\mathrm{Lap}}, \lambda_0$ and the properties ($\star\star$), we see

$$\left(\frac{2}{r^2} - Cr^{-3} \right) \int_{\Sigma_r} |w - \overline{w}|^2\, d\mu \leq Cr^{-4} \int_{\Sigma_r} \left(|w - \overline{w}|^2 + |\overline{w}||w - \overline{w}| \right) d\mu.$$

From the elementary inequality $|\overline{w}||w - \overline{w}| \leq \epsilon r^2 |w - \overline{w}|^2 + C(\epsilon) r^{-2} \overline{w}^2$, we obtain

$$\|w - \overline{w}\|_{L^2(\Sigma_r)} \leq Cr^{-2} |\overline{w}| |\Sigma_r|^{\frac{1}{2}}. \tag{5.3}$$

In particular, if w is not constant, then $\overline{w} \neq 0$.

Let u be an eigenfunction with respect to λ_1. Then

$$L_{\Sigma_r}(u - \overline{u}) = \lambda_1 (u - \overline{u}) + (\lambda_1 + |A|^2 + \mathrm{Ric}(\nu,\nu)) \overline{u}.$$

We multiply the above identity by $(u - \overline{u})$ and integrate over Σ_r. Since $u - \overline{u}$ has zero mean value, we apply Theorem 5.2 and obtain

$$\begin{aligned}\left(\frac{6m}{r^3} - Cr^{-4} \right) \int_{\Sigma_r} |u - \overline{u}|^2\, d\mu &\leq \int_{\Sigma_r} (u - \overline{u}) L_{\Sigma_r} (u - \overline{u})\, d\mu \\ &= \lambda_1 \int_{\Sigma_r} (u - \overline{u})^2\, d\mu + \int_{\Sigma_r} (\lambda_1 + |A|^2 + \mathrm{Ric}(\nu,\nu)) \overline{u} (u - \overline{u})\, d\mu \\ &\leq \lambda_1 \int_{\Sigma_r} (u - \overline{u})^2\, d\mu + Cr^{-4} \int_{\Sigma_r} |\overline{u}||u - \overline{u}|\, d\mu \\ &\leq \lambda_1 \int_{\Sigma_r} (u - \overline{u})^2\, d\mu + Cr^{-4} |\overline{u}| |\Sigma_r|^{1/2} \|u - \overline{u}\|_{L^2}.\end{aligned}$$

To estimate the last term on the right hand side, we note that

$$0 = \int_{\Sigma_r} uw\, d\mu = \int_{\Sigma_r} (u - \overline{u})(w - \overline{w})\, d\mu + \int_{\Sigma_r} u\overline{w}\, d\mu.$$

By the Hölder inequality and the L^2 bound of $(w - \overline{w})$ in (5.3),

$$|\overline{u}||\Sigma_r| = \left| \int_{\Sigma_r} u\, d\mu \right| \leq |\overline{w}|^{-1} \|u - \overline{u}\|_{L^2} \|w - \overline{w}\|_{L^2} \leq Cr^{-2} |\Sigma_r|^{1/2} \|u - \overline{u}\|_{L^2}.$$

Putting the above inequalities together, we have

$$\left(\frac{6m}{r^3} - Cr^{-4} \right) \|u - \overline{u}\|_{L^2}^2 \leq \lambda_1 \|u - \overline{u}\|_{L^2}^2 + Cr^{-6} \|u - \overline{u}\|_{L^2}^2.$$

It implies that

$$\lambda_1 \geq \frac{6m}{r^3} - Cr^{-4}.$$

This implies that if $m > 0$, for r sufficiently large, $L_{\Sigma_r} : C^{2,\alpha}(\Sigma_r) \to C^{0,\alpha}(\Sigma_r)$ is injective. By Fredholm alternative, L_{Σ_r} is surjective. Hence, it is a linear isomorphism. □

5.3. **Foliations.** Let (M, g) be asymptotic to Schwarzschild of $m > 0$. Let Σ_r be a surface in the family $\{\Sigma_r\}$ that satisfy the properties $(\star)$ and $(\star\star)$. As before, we define the mean curvature operator $\mathcal{H} : C^{2,\alpha}(\Sigma_r) \to C^{0,\alpha}(\Sigma_r)$ to be the differential operator that maps ϕ to the mean curvature of the normal graph $\{x + \nu\phi : x \in \Sigma_r\}$. Theorem 5.3 says that the linearized operator $d\mathcal{H} = L_{\Sigma_r}$ is a linear isomorphism for r sufficiently large. Recall the Inverse Function Theorem:

Theorem 5.4 (Inverse Function Theorem)**.** *Let E and F be Banach spaces, and let U be an open subset of E. Suppose $f : U \subset E \to F$ is of class C^k, $k \geq 1$. Let $x_0 \in U$. Suppose that $Df(x_0)$ is a linear isomorphism. Then f is a C^k diffeomorphism of some neighborhood of x_0 onto some neighborhood of $f(x_0)$.*

Fix r such that $d\mathcal{H}$ is a linear isomorphism on the surface Σ_r of constant mean curvature h_0. The Inverse Function Theorem implies that there exists $\epsilon > 0$ such that for each constant $h \in (h_0 - \epsilon, h_0 + \epsilon)$, there is the *unique* normal graph over Σ that has constant mean curvature h. This also implies that we can define a differentiable deformation $F : \Sigma_r \times (h_0 - \epsilon, h_0 + \epsilon) \to M$ by sending (Σ_r, h) to the unique normal graph over Σ_r that has constant mean curvature h. Let $H(h) = h$ denote the mean curvature of the normal graph $F(\Sigma_r, h)$. Since each surface has constant mean curvature, only the normal component $\frac{\partial}{\partial h} F$ contributes to the evolution of $H(h)$. Thus,

$$1 = \left. \frac{d}{dh} \right|_{h=h_0} H(h) = L_{\Sigma_r} \phi, \tag{5.4}$$

where $\phi = g\left(\left. \frac{\partial}{\partial h} \right|_{h=h_0} F, \nu \right)$. In the following we show that ϕ has a sign, from which it follows that members of the family of constant mean curvature surfaces do not intersect, and in fact form a foliation.

We recall below a standard application of Moser iteration and include the proof since our setting is slightly different from [26, Theorem 8.17].

Proposition 5.5. *Let (Σ, g) be a two-dimensional closed Riemannian manifold. Let v be a C^2 solution to*

$$-\Delta_\Sigma v - Qv = f, \tag{5.5}$$

where $Q, f \in L^\infty(\Sigma)$ and $Q \geq 0$. Then, for any $p_0 \geq 2$,

$$\sup_\Sigma |v| \leq C_0(\|v\|_{L^{p_0}(\Sigma)} + k),$$

where $k = \|f\|_{L^\infty}$ and C_0 depends on $p_0, \Sigma, g, \|Q\|_{L^\infty}$.

Proof. Let $v^+ := \max_\Sigma\{v, 0\}$. Note $v^+ \in W^{1,2}(\Sigma)$ and

$$\nabla v^+ = \begin{cases} \nabla v & \text{if } v > 0 \\ 0 & \text{if } v \leq 0 \end{cases}.$$

Let $w = v^+ + k$, where k is defined as in the proposition. For any real number $p \geq 1$, we multiply the differential equation (5.5) by w^p and integrate over Σ:

$$\begin{aligned}\int_\Sigma |\nabla(w^{\frac{p+1}{2}})|^2 \, d\mu &= \frac{(p+1)^2}{4p} \int_\Sigma (Qvw^p + fw^p) \, d\mu \\ &\leq p \int_\Sigma (Q+1) w^{p+1} \, d\mu \\ &\leq p \max_\Sigma (Q+1) \int_\Sigma w^{p+1} \, d\mu,\end{aligned} \tag{5.6}$$

where in the first inequality we use the assumption that $Q \geq 0$. The above computation establishes an upper bound of the $W^{1,2}$-norm of $w^{\frac{p+1}{2}}$ by purely the L^2-norm of $w^{\frac{p+1}{2}}$. To begin the iteration procedure, we need to relate the higher order L^q-norm to the $W^{1,2}$-norm. For manifolds of higher dimensions, the standard procedure is to apply the Gagliardo-Nirenberg-Sobolev inequality: for $1 \leq p < n$,

$$\|u\|_{L^{p^*}} \leq C_0 \|u\|_{W^{1,p}},$$

where $p^* = \frac{np}{n-p}$ and n is the dimension of the manifold. However, in our case $n = 2$ and $p = 2$ is the borderline case of the Gagliardo-Nirenberg-Sobolev inequality, so we use another inequality specifically for a two-dimensional manifold Σ: for any $1 \leq q < \infty$,

$$\|u\|_{L^q(\Sigma)} \leq C_0 \sqrt{q} \|u\|_{W^{1,2}(\Sigma)}, \tag{5.7}$$

where C_0 depends on (Σ, g).

Substitute u in (5.7) with $w^{\frac{p+1}{2}}$, and let $q = 2\kappa > 2$ be a fixed real number. Together with (5.6) and enlarging C_0 if necessary, we obtain, for any $1 \leq p < \infty$,

$$\|w\|_{L^{(p+1)\kappa}} \leq C_0^{\frac{1}{p+1}} (p+1)^{\frac{1}{p+1}} \|w\|_{L^{p+1}}, \tag{5.8}$$

where C_0 depends on $\kappa, \Sigma, g, \|Q\|_{L^\infty}$. Now we define a sequence of numbers

$$p_0 = p + 1, \quad p_i = p_{(i-1)}\kappa = (p+1)\kappa^i, \quad i = 1, 2, \ldots.$$

The estimate (5.8) implies that

$$\|w\|_{L^{p_{(i+1)}}} \leq C_0^{\sum_{j=0}^i \frac{1}{p_j}} \prod_{j=0}^i p_j^{\frac{1}{p_j}} \|w\|_{L^{p_0}}.$$

Note that as i tending to infinity $\|w\|_{L^{p_{i+1}}(\Sigma)}$ converges to $\|w\|_{L^\infty}$ and the coefficient on the right hand side converges because $\kappa > 1$. This implies that, for κ fixed and for any $p_0 \geq 2$,

$$v \leq \sup_\Sigma w \leq C_0 \|w\|_{L^{p_0}} \leq C_0 \left(\|v^+\|_{L^{p_0}} + k \right) \leq C_0 \left(\|v\|_{L^{p_0}} + k \right),$$

where C_0 depends on $p_0, \Sigma, \|Q\|_{L^\infty}$. Substituting v with $-v$ yields

$$-v \leq C_0 \left(\|v\|_{L^{p_0}} + k \right).$$

The desired estimate follows. □

We now use a scaling argument to factor out the dependence of the constant C_0 in Proposition 5.5 from the family of surfaces $\{\Sigma_r\}$.

Proposition 5.6. *Let (M, g) be asymptotic to Schwarzschild of mass m. Suppose $\{\Sigma_r\}$ is a family of surfaces satisfying the property $(\star)$. For each r, let v be a C^2 solution to*

$$-\Delta_{\Sigma_r} v - Qv = f,$$

where $Q, f \in L^\infty(\Sigma_r)$ and $Q \geq 0$. Then, for any $p_0 \geq 2$,

$$\sup_{\Sigma_r} |v| \leq C_0 (r^{-\frac{2}{p_0}} \|v\|_{L^{p_0}(\Sigma_r)} + k),$$

where $k = \|f\|_{L^\infty}$ and C_0 depends on $g, p_0, \|Q\|_{L^\infty}$ (but independent of r).

Proof. The property $(\star)$ gives a family of smooth diffeomeomorphisms $F_r : \mathbb{S}^2 \to \Sigma_r$ such that $|dF_r - r\mathrm{Id}| = O_2(1)$, where Id is the identification map from $T\mathbb{S}^2$ to $T\Sigma_r$. It implies the pull back metric satisfies

$$\|F_r^* g_{\Sigma_r} - g_{\mathbb{S}^2}\|_{C^2} \leq C,$$

where C depends only on g. Considering the pull-back of the differential equations for $v \circ F_r$ and applying Proposition 5.5 on the fixed geometry $(\mathbb{S}^2, g_{\mathbb{S}^2})$, we obtain

$$\sup_{\mathbb{S}^2} |v \circ F_r| \leq C_0 (\|v \circ F_r\|_{L^{p_0}(\mathbb{S}^2)} + k),$$

where C_0 depends on g, p_0, $k = \|f \circ F_r\|_{L^\infty(\mathbb{S}^2)}$, and Q. Using the area formula for the L^{p_0}-norm, we have the desired estimate. □

Theorem 5.7. *Let (M, g) be asymptotic to Schwarzschild of mass $m > 0$. Suppose $\{\Sigma_r\}$ is a family of surfaces satisfying the properties $(\star)$ and $(\star\star)$. Let $u \in C^{2,\alpha}(\Sigma_r)$ satisfy*

$$L_{\Sigma_r} u := -\Delta_{\Sigma_r} u - (|A|^2 + \mathrm{Ric}(\nu, \nu)) u = c$$

for some constant c. Then there exists r_0 sufficiently large such that for each $r \geq r_0$,

$$\sup_{\Sigma_r} |u - \overline{u}| \leq C r^{-1} |\overline{u}|,$$

where C depends only on g. As a consequence, for r_0 sufficiently large, the solution u is either positive or negative for each $r \geq r_0$.

Proof. Note that $(u-\overline{u})$ satisfies the equation $L_{\Sigma_r}(u-\overline{u}) = c + (|A|^2 + \mathrm{Ric}(\nu,\nu))\overline{u}$. By Theorem 5.2 and the estimate on $|A|^2 + \mathrm{Ric}(\nu,\nu)$ from $(\star\star)$, we obtain

$$\begin{aligned}\left(\frac{6m}{r^3} - Cr^{-4}\right)\int_{\Sigma_r}|u-\overline{u}|^2\,d\mu &\le \int_{\Sigma_r}(u-\overline{u})L(u-\overline{u})\,d\mu\\ &= \int_{\Sigma_r}(|A|^2+\mathrm{Ric}(\nu,\nu))\overline{u}(u-\overline{u})\,d\mu\\ &\le Cr^{-4}\int_{\Sigma_r}|\overline{u}||u-\overline{u}|\,d\mu\\ &\le Cr^{-4}\left(\int_{\Sigma_r}|u-\overline{u}|^2\,d\mu\right)^{\frac12}\left(\int_{\Sigma_r}|\overline{u}|^2\,d\mu\right)^{\frac12}.\end{aligned}$$

This implies

$$\|u-\overline{u}\|_{L^2(\Sigma_r)} \le Cm^{-1}r^{-1}|\overline{u}||\Sigma_r|^{\frac12}.$$

By Proposition 5.6,

$$\sup_{\Sigma_r}|u-\overline{u}| \le C(r^{-1}\|u-\overline{u}\|_{L^2(\Sigma_r)} + k),$$

where $k = \max_{\Sigma_r}|c + (|A|^2+\mathrm{Ric}(\nu,\nu))\overline{u}|$ and C depends on $\sup_{\Sigma_r}(|A|^2+\mathrm{Ric}(\nu,\nu))$. To estimate the constant c, we integrate $L_{\Sigma_r}u = c$ over Σ_r and obtain

$$\begin{aligned}|\Sigma_r||c| &\le \left|\int_{\Sigma_r}(|A|^2+\mathrm{Ric}(\nu,\nu))(u-\overline{u})\,d\mu\right| + \left|\int_{\Sigma_r}(|A|^2+\mathrm{Ric}(\nu,\nu))\overline{u}\,d\mu\right|\\ &\le C(r^{-4}\|u-\overline{u}\|_{L^2}|\Sigma_r|^{\frac12} + r^{-4}|\Sigma_r||\overline{u}|).\end{aligned}$$

By the above estimates and the properties $(\star\star)$, we have

$$\sup_{\Sigma_r}|u-\overline{u}| \le Cr^{-1}|\overline{u}|.$$

□

Applying Theorem 5.7 to (5.4), we obtain the the following result.

Corollary 5.8. *Let (M,g) be asymptotic to Schwarzschild of mass $m>0$. Suppose $\{\Sigma_r\}$ is a family of surfaces satisfying the properties $(\star)$ and $(\star\star)$. Then there exists $r_0>0$ such that the family of surfaces $\{\Sigma_r\}$ for $r \ge r_0$ form a foliation.*

6. Density Theorems

6.1. **Weighted Sobolev spaces.** We introduce a topology on the space of asymptotically flat initial data sets using the following weighted norm. Let B be a ball in $\mathbb{R}^n$ centered at the origin. For $k \in \{0,1,\dots\}$, $p \ge 0$, and $q \in \mathbb{R}$, we define the **weighted Sobolev space** $W^{k,p}_{-q}(\mathbb{R}^n \setminus B)$ to be the set of functions $f \in W^{k,p}_{-q}(\mathbb{R}^n \setminus B)$ with

$$\|f\|_{W^{k,p}_{-q}(\mathbb{R}^n\setminus B)} := \left(\int_{\mathbb{R}^n\setminus B}\sum_{|I|\le k}\left(\left|\partial^I f(x)\right||x|^{|I|+q}\right)^p|x|^{-n}\,dx\right)^{\frac1p} < \infty.$$

When $p = \infty$,

$$\|f\|_{W^{k,\infty}_{-q}(\mathbb{R}^n\setminus B)} := \sum_{|I|\le k} \operatorname*{ess\,sup}_{\mathbb{R}^n\setminus B} |\partial^I f||x|^{|I|+q}.$$

Suppose M is a smooth manifold such that there is a compact set $K \subset M$ and a diffeomorphism $M \setminus K \cong \mathbb{R}^n \setminus B$. Choose an atlas for M that consists of the diffeomorphism $M \setminus K \cong \mathbb{R}^n \setminus B$ and finitely many precompact charts on K. We define the $W^{k,p}_{-q}(M)$ norm on M by summing over the $W^{k,p}_{-q}$ norm on the noncompact chart and the $W^{k,p}$ norm on the precompact charts. The definition extends to the tensor bundles of M by considering the components with respect to these charts, and can also easily extend to an asymptotically flat manifold with a finite number of ends. We sometimes write $W^{k,p}_{-q}$ for $W^{k,p}_{-q}(M)$.

It is known that the ADM energy and linear momentum are continuous functions with respect to the appropriate weighted Sobolev topology.

Theorem 6.1. *Let $p > n \ge 3$, $q \in \left(\frac{n-2}{2}, n-2\right)$, $q_0 > 0$. Let (g,k) and $(\bar g, \bar k)$ be $C^2_{\mathrm{loc}} \times C^1_{\mathrm{loc}}$ asymptotically flat initial data sets such that*

$$(g - g_0, k), (\bar g - g_0, \bar k) \in W^{2,p}_{-q} \times W^{1,p}_{-1-q},$$

where g_0 is a smooth symmetric $(0,2)$ tensor that coincides with $g_{\mathbb{E}}$ on $M \setminus K$, and such that

$$\mu, J, \bar\mu, \bar J \in W^{0,p}_{-n-q_0}.$$

Let $\epsilon > 0$. There exists $\delta > 0$ such that if

$$\|g - \bar g\|_{W^{2,p}_{-q}} \le \delta \quad \text{and} \quad \|k - \bar k\|_{W^{1,p}_{-1-q}} \le \delta,$$

then

$$|E - \bar E| < \epsilon \quad \text{and} \quad |P - \bar P| < \epsilon.$$

The proof of this fact goes back to [45, p. 50] for E only and to [18, p. 198] in the vacuum case. The proof of the general case can be found in [30, Proposition 2.4] and [21, Proposition 19].

On the other hand, the BORT center of mass and the ADM angular momentum may not be defined in general for asymptotically flat initial data sets since the integrals (1.1) and (1.2) may diverge [28, 12, 10, 11]. In fact, the BORT center of mass and the ADM angular momentum are *discontinuous* with respect to above topology (see Theorem 6.7 below). Nevertheless, if we consider a topology that incorporates the Regge-Teitelboim conditions, we have an analogous continuity result for the center of mass and angular momentum. In the following we denote $f^{\mathrm{odd}}(x) = (f(x) - f(-x))/2$ and $f^{\mathrm{even}}(x) = (f(x) + f(-x))/2$ with respect to an asymptotically flat coordinate chart.

Theorem 6.2 (cf. Huang [30, Proposition 2.4], [27, Theorem 2.2])**.** *Let $p > n \ge 3$, $q \in \left(\frac{n-2}{2}, n-2\right)$. Let $(g,k), (\bar g, \bar k), (\mu, J), (\bar\mu, \bar J)$ satisfy the assumptions in Theorem 6.1. Suppose they also satisfy*

$$(g^{\mathrm{odd}}_{ij}, k^{\mathrm{even}}_{ij}), (\bar g^{\mathrm{odd}}_{ij}, \bar k^{\mathrm{even}}_{ij}) \in W^{2,p}_{-1-q}(M \setminus K) \times W^{1,p}_{-2-q}(M \setminus K)$$

and

$$\mu^{\mathrm{odd}}, J^{\mathrm{odd}}_i, \bar\mu^{\mathrm{odd}}, \bar J^{\mathrm{odd}}_i \in W^{0,p}_{-n-q_0-1}(M \setminus K).$$

Let $\epsilon > 0$. There exists $\delta > 0$ such that if

$$\|g^{\mathrm{odd}} - \bar{g}^{\mathrm{odd}}\|_{W^{2,p}_{-1-q}(M\setminus K)} \le \delta \quad \textit{and} \quad \|k^{\mathrm{even}} - \bar{k}^{\mathrm{even}}\|_{W^{1,p}_{-2-q}(M\setminus K)} \le \delta$$

then

$$|\mathcal{C}_{\mathrm{BORT}} - \overline{\mathcal{C}}_{\mathrm{BORT}}| < \epsilon \quad \textit{and} \quad |\mathcal{J} - \overline{\mathcal{J}}| < \epsilon.$$

6.2. **Scalar curvature equation.** We discuss a density result for the scalar curvature equation due to Schoen and Yau. The density argument is used in the proof of the Riemannian positive mass theorem and enables them to reduce the case of the general asymptotically flat metrics to the case that the metrics are scalar flat and conformally flat at infinity. In what follows, we consider an asymptotically flat manifold M of dimension $n \ge 3$.

Theorem 6.3 (Schoen-Yau [45])**.** *Let (M, g) be an n-dimensional asymptotically flat initial data set with nonnegative scalar curvature and the ADM mass m. Given $\epsilon > 0$, there exists an asymptotically flat metric $\overline{g}$ with zero scalar curvature such that, outside a compact set of M, the metric has the form*

$$\overline{g}_{ij} = u^{\frac{4}{n-2}}\delta_{ij}$$

with $u = 1 + \frac{\overline{m}}{2} r^{2-n} + O(r^{1-n})$, where $\overline{m}$ is the ADM mass of $\overline{g}$ and

$$\overline{m} \le m + \epsilon.$$

Lemma 6.4 (Schoen-Yau [44, Lemma 3.3])**.** *Let (M, g) be an n-dimensional asymptotically flat initial data set with the ADM mass m. Suppose the scalar curvature $R(g) \ge 0$ is positive somewhere, and $R(g) = O(|x|^{-3-q_0})$ for some $q_0 > 1$. Then there exist constant $A < 0$ and a unique metric $\overline{g} = u^{\frac{4}{n-2}} g$ with zero scalar curvature such that*

$$u = 1 + \frac{A}{2} r^{2-n} + O(r^{1-n}).$$

Furthermore, the ADM mass $\overline{m}$ of $\bar{g}$ satisfies $\overline{m} = m + A < m$.

Proof. Let $\overline{g} = u^{\frac{4}{n-2}} g$. The scalar curvatures of g and $\bar{g}$ are related by

$$R(\overline{g}) = u^{-\frac{n+2}{n-2}} \left(R(g) u - \frac{4(n-1)}{n-2} \Delta_g u \right).$$

Denote the conformal Laplace operator by $L = \Delta_g - \frac{n-2}{4(n-1)} R(g)$. Let $p > n$, $q \in \left(\frac{n-2}{2}, n-2\right)$. By [4, Proposition 1.14], $L : W^{2,p}_{-q} \to W^{0,p}_{-2-q}$ is a Fredholm operator of index zero. To find a solution to the inhomogeneous equation $Lv = f$ for $f \in W^{0,p}_{-2-q}$, it suffices to prove that L has a trivial kernel by the Fredholm alternative. Let $v \in W^{2,p}_{-q}$ satisfy $Lv = 0$. Multiplying the equation $Lv = 0$ by v and applying the divergence theorem, we have

$$\begin{aligned} 0 \le \int_M |\nabla v|^2 \, d\sigma &= -\frac{n-2}{4(n-1)} \int_M R(g) v^2 \, d\sigma + \lim_{r\to\infty} \int_{|x|=r} v \frac{\partial v}{\partial \nu} \, d\mu \\ &= -\frac{n-2}{4(n-1)} \int_M R(g) v^2 \, d\sigma \le 0, \end{aligned}$$

where we use the fall-off rates of $v, \partial v$ to compute the boundary term. We conclude that $v \equiv 0$. Therefore there is a unique solution to $Lv = \frac{n-2}{4(n-1)} R(g)$. Let $u = v + 1$. Then u satisfies $Lu = 0$. Note that $u > 0$ everywhere by the strong maximum principle. Then

$\overline{g} = u^{\frac{4}{n-2}} g$ is the desired metric. The asymptotic expansion of u follows from [4, Theorem 1.17].

To show that $A < 0$, we integrate $Lu = 0$ over a large ball and apply the divergence theorem:

$$\begin{aligned} 0 < \lim_{r\to\infty} \frac{n-2}{4(n-1)} \int_{|x|\le r} R(g) u \, d\sigma &= \lim_{r\to\infty} \int_{|x|=r} \frac{\partial u}{\partial \nu} \, d\mu \\ &= \lim_{r\to\infty} \int_{|x|=r} -\frac{(n-2)A}{2} r^{1-n} \, d\mu \\ &= -\frac{n-2}{2} \omega_{n-1} A. \end{aligned}$$

□

Proof of Theorem 6.3. By Lemma 6.4, we may assume without loss of generality that g has zero scalar curvature. For $\lambda \ge 1$ large, we define the cut-off metric

$$\hat{g}_\lambda := \chi_\lambda g + (1 - \chi_\lambda) g_{\mathbb{E}},$$

where $\chi_\lambda(x) = \chi(x/\lambda)$ and χ is a smooth cut-off function on $\mathbb{R}^n$ that is 1 on $\{|x| \le 1\}$ and 0 on $\{|x| \ge 2\}$. Note that the cut-off metric has zero scalar curvature everywhere except the interpolating region $\lambda \le |x| \le 2\lambda$ and $R(\hat{g}_\lambda) = O(\lambda^{-n})$ there. We would like to find a metric $\bar{g} = u^{\frac{4}{n-2}} \hat{g}_\lambda$ with zero scalar curvature in the conformal class of $\hat{g}_\lambda$. By the transformation formula of the scalar curvature, it suffices to find a positive function u that tends to 1 at infinity and satisfies

$$\Delta_{\hat{g}_\lambda} u - \frac{n-2}{4(n-1)} R(\hat{g}_\lambda) u = 0.$$

Note that $R(\hat{g}_\lambda)$ may not be nonnegative everywhere, so the proof of Lemma 6.4 cannot be applied. The approach of Schoen and Yau relies on a Sobolev inequality and requires $\|R(\hat{g}_\lambda)^-\|_{L^{\frac{n}{2}}(M)}$ sufficiently small, which is achieved by choosing λ sufficiently large. We refer the details to [44, Lemma 3.2].

The last statement that $\overline{m} \le m + \epsilon$ follows from the continuity of the ADM mass by Theorem 6.1. □

6.3. **Einstein constraint equations.** Let (M, g, k) be an initial data set. Define the momentum tensor

$$\pi = k - (\mathrm{tr}_g k) g.$$

It is often convenient to express initial data in terms of π rather than k. We will refer to (M, g, π) as an initial data set in this section and define the constraint map

$$\Phi(g, \pi) = (2\mu, J) = \left(R(g) - |\pi|_g^2 + \tfrac{1}{n-1} (\mathrm{tr}_g \pi)^2, \mathrm{div}_g \pi \right).$$

We say that (M, g, π) has **harmonic asymptotics** if there exist a smooth function u and a smooth vector field X such that $u \to 1, X \to 0$ at infinity and, outside a compact set of M,

$$\begin{aligned} g &= u^{\frac{4}{n-2}} g_{\mathbb{E}} \\ \pi &= u^{\frac{2}{n-2}} (\mathcal{L}_{g_{\mathbb{E}}} X), \end{aligned}$$

where the operator $\mathcal{L}_g$ is defined by $\mathcal{L}_g X = L_X g - \mathrm{div}_g(X) g$ and $L_X g$ is the Lie derivative. Throughout this section, we denote by g_0 a smooth symmetric $(0,2)$ tensor on M that coincides with $g_{\mathbb{E}}$ on $M \setminus K$.

The term "harmonic" follows from the following proposition that the leading order terms of the function u and the vector field X are harmonic.

Proposition 6.5 (Corvino-Schoen [18], see also [21, Proposition 24]). *Let $p > n$, $q \in (\frac{n-2}{2}, n-2)$, $q_0 > 1$ (rather than just $q_0 > 0$). Suppose that (M^n, g, π) is an asymptotically flat initial data set that satisfies*

$$(g - g_0, \pi) \in W^{2,p}_{-q}(M) \times W^{1,p}_{-1-q}(M),$$

and

$$(\mu, J) \in W^{0,p}_{-n-q_0}.$$

such that (g, π) has harmonic asymptotics:

$$g = u^{\frac{4}{n-2}} g_{\mathbb{E}}, \quad \pi = u^{\frac{2}{n-2}} \mathcal{L}_{g_{\mathbb{E}}} X, \tag{6.1}$$

outside a compact set, for some $(u-1, X) \in W^{2,p}_{-q}$. Then (u, X) admits an expansion

$$u(x) = 1 + a|x|^{2-n} + O(|x|^{1-n}) \qquad \text{and} \qquad X_i(x) = b_i |x|^{2-n} + O(|x|^{1-n}),$$

where $X = X^i \frac{\partial}{\partial x^i}$.

A generalization of Theorem 6.3 to the full constraint equations is the following result, that initial data sets with harmonic asymptotics are dense among general asymptotically flat initial data set.

Theorem 6.6 (Corvino-Schoen [18, Theorem 1]). *Let $p > n$, $q \in (\frac{n-2}{2}, n-2)$. Let (g, π) and $(\bar{g}, \bar{\pi})$ be vacuum asymptotically flat initial data sets*

$$(g - g_0, \pi) \in W^{2,p}_{-q} \times W^{1,p}_{-1-q}.$$

Let $\epsilon > 0$. There exists a vacuum asymptotically flat initial data set $(\bar{g}, \bar{\pi})$ with harmonic asymptotics such that

$$\|g - \bar{g}\|_{W^{2,p}_{-q}} \leq \epsilon, \quad \|\pi - \bar{\pi}\|_{W^{1,p}_{-1-q}} \leq \epsilon$$

and

$$|E - \bar{E}| < \epsilon, \quad |P - \bar{P}| < \epsilon.$$

Proof. For $\lambda \geq 1$ large define the cut-off initial data

$$\begin{aligned}(\hat{g}_\lambda)_{ij} &= \chi_\lambda g_{ij} + (1 - \chi_\lambda)\delta_{ij}\\ \hat{\pi}_\lambda &= \chi_\lambda \pi,\end{aligned}$$

where $\chi_\lambda(x) = \chi(x/\lambda)$ and χ is a smooth cut-off function on $\mathbb{R}^n$ such that χ is 1 on $\{|x| \leq 1\}$ and 0 on $\{|x| \geq 2\}$. In the following, we suppress the subscript λ when the context is clear.

The system of the Einstein constraint equations forms an underdetermined system: the number of unknowns is greater than the number of the equations that determine them.

One reason to introduce a function u and a vector field X in the expression of harmonic asymptotics is to obtain a well-determined elliptic system. Let

$$\tilde{g} = u^{\frac{4}{n-2}}\hat{g}$$
$$\tilde{\pi} = u^{\frac{2}{n-2}}(\hat{\pi} + \mathcal{L}_{\hat{g}}X).$$

Now we would like to find u tending to 1 and X tending to 0 at infinity such that $(\tilde{g},\tilde{\pi})$ satisfies the vacuum constraints.

Define $T_{(\hat{g},\hat{\pi})} : (W^{2,p}_{-q}+1)\times W^{2,p}_{-q} \to W^{0,p}_{-2-q}$ to be the constraint map $T_{(\hat{g},\hat{\pi})}(u,X) = \Phi(\tilde{g},\tilde{\pi})$. The map $T_{(g,\pi)} : (W^{2,p}_{-q}+1)\times W^{2,p}_{-q} \to W^{0,p}_{-2-q}$ is defined analogously. Note that $T_{(\hat{g},\hat{\pi})}$ and $T_{(g,\pi)}$ are smooth maps. The linearization of $T_{(\hat{g},\hat{\pi})}$ at $(1,0)$ is

$$\begin{aligned}
&DT_{(\hat{g},\hat{\pi})}|_{(1,0)}(v,Z)\\
&= \Bigg(-\frac{4(n-1)}{n-2}\Delta_{\hat{g}}v - \frac{4(n-1)}{n-2}\left[R_{\hat{g}} - |\hat{\pi}|^2_{\hat{g}} + \tfrac{1}{n-1}(\mathrm{tr}_{\hat{g}}\hat{\pi})^2\right]v - 4Z_{k;\ell}\hat{\pi}^{k\ell} - \tfrac{2}{n-1}\mathrm{tr}_{\hat{g}}\hat{\pi}\,\mathrm{div}_{\hat{g}}Z,\\
&\qquad \mathrm{div}_{\hat{g}}(\mathcal{L}_{\hat{g}}Z)_j + \frac{2(n-1)}{n-2}v_{,k}\hat{\pi}^k_j - \frac{2}{n-2}v_{,j}\mathrm{tr}_{\hat{g}}\hat{\pi} - \frac{2}{n-2}(\mathrm{div}_{\hat{g}}\hat{\pi})_j v\Bigg),
\end{aligned}$$

where indices are raised and covariant derivatives are taken with respect to $\hat{g}$. Because $q \in (\frac{n-2}{2}, n-2)$ and $p > n$, $DT_{(\hat{g},\hat{\pi})}|_{(1,0)}$ and $DT_{(g,\pi)}|_{(1,0)}$ are Fredholm operators of index 0 for λ sufficiently large [4]. Instead of proving the linearization has a trivial kernel as in the proof of Theorem 6.3 which seems difficult for the system, we use the following argument.

Let K_1 be a complementing subspace for the kernel of $DT_{(g,\pi)}|_{(1,0)}$ in $W^{2,p}_{-q}\times W^{1,p}_{-1-q}$. Since the linearization $D\Phi|_{(g,\pi)} : W^{2,p}_{-q}\times W^{1,p}_{-1-q} \to W^{0,p}_{-2-q}$ is surjective [18, Proposition 3.1], and because $DT_{(g,\pi)}|_{(1,0)}$ is Fredholm we can find smooth compactly supported symmetric $(0,2)$-tensors $\{(h_k,w_k)\}_{k=1}^N$ whose images $\{D\Phi|_{(g,\pi)}(h_k,w_k)\}$ form a basis for a complementing subspace of the image of $DT_{(g,\pi)}|_{(1,0)}$ in $W^{0,p}_{-2-q}$. Let $K_2 = \mathrm{span}\{(h_k,w_k)\}_{k=1}^N$. For $(u-1,X)\in K_1$ and $(h,w)\in K_2$, define the maps $\overline{T}_{(\hat{g},\hat{\pi})}, \overline{T}_{(g,\pi)}$ as follows:

$$\overline{T}_{(\hat{g},\hat{\pi})}(u,X,h,w) = \Phi(u^{\frac{4}{n-2}}\hat{g} + h, u^{\frac{2}{n-2}}(\hat{\pi} + \mathcal{L}_{\hat{g}}X) + w)$$
$$\overline{T}_{(g,\pi)}(u,X,h,w) = \Phi(u^{\frac{4}{n-2}}g + h, u^{\frac{2}{n-2}}(\pi + \mathcal{L}_{g}X) + w).$$

Observe that $D\overline{T}_{(\hat{g},\hat{\pi})}|_{(1,0,0,0)}$ is an isomorphism for λ sufficiently large by construction.

Using that $(\hat{g},\hat{\pi})$ converges to (g,π) in $W^{2,p}_{-q}\times W^{1,p}_{-q-1}$ as $\lambda\to\infty$, it is easy to see that $D\overline{T}_{(\hat{g},\hat{\pi})}|_{(u,X,h,w)}$ converges to $D\overline{T}_{(g,\pi)}|_{(u,X,h,w)}$ as $\lambda\to\infty$, locally uniformly in (u,X,h,w) in the strong operator topology. By the inverse function theorem, for all $\lambda\geq 1$ sufficiently large, $\overline{T}_{(\hat{g},\hat{\pi})}$ restricts to a diffeomorphism defined on an open neighborhood of $(1,0,0,0)$ (independent of $\lambda\geq 1$) and onto an open neighborhood containing a ball centered at $(0,0)$ in $W^{0,p}_{-2-q}$. The preimage $\overline{T}^{-1}_{(\hat{g},\hat{\pi})}(0,0)$ gives the desired solutions.

The inequalities

$$|E-\bar{E}|<\epsilon, \quad |P-\bar{P}|<\epsilon$$

follow from Theorem 6.1. □

The vacuum assumption in the above theorem can be replaced by appropriate assumptions on (μ,J). In fact, using a more delicate perturbation argument, one can prove that if (g,π) satisfies the dominant energy condition, it is possible to obtain a strict dominant energy condition for the approximate data $(\bar{g}, \bar{\pi})$ with harmonic asymptotics [21, Theorem 18]. This fact is used in the proof of the spacetime positive mass theorem to reduce the general case of the theorem to the special case of initial data that has harmonic asymptotics with a strict dominant energy condition [21].

6.4. **Applications to the center of mass and angular momentum.** Generalizing the proof of Theorem 6.6, we show that one can arbitrarily specify the BORT center of mass and the ADM angular momentum.

Theorem 6.7 (Huang-Schoen-Wang [31, Theorem 3])**.** *Let (M, g, π) be a three-dimensional vacuum asymptotically flat initial data set satisfying the Regge-Teitelboim conditions and $E > |P|$. Given any constant vectors $\vec{\alpha}_0, \vec{\gamma}_0 \in \mathbb{R}^3$, there exist $\epsilon > 0$ and a vacuum initial data set $(\bar{g}, \bar{\pi})$ satisfying the Regge-Teitelboim conditions such that*

$$\|g - \bar{g}\|_{W^{2,p}_{-q}} \le \epsilon, \quad \|\pi - \bar{\pi}\|_{W^{1,p}_{-1-q}} \le \epsilon,$$

and

$$\overline{E} = E, \quad \overline{P} = P, \quad \overline{\mathcal{J}} = \mathcal{J} + \vec{\alpha}_0, \quad \overline{\mathcal{C}}_{\mathrm{BORT}} = \mathcal{C}_{\mathrm{BORT}} + \vec{\gamma}_0.$$

Analogous to the positive mass conjecture, there is a conjectured inequality between the ADM energy and angular momentum. It has been proven that $E \ge \sqrt{|\mathcal{J}|}$ for axial-symmetric asymptotically flat black hole initial data sets [19, 15, 17, 16, 47, 53]. See also [52]. However, Theorem 6.7 implies that such inequality does not hold in general for asymptotically flat data sets without axial-symmetry.

The proof of Theorem 6.7 is essentially along the same line as the proof of Theorem 6.6, but different cutoff data sets are employed. Let σ, τ be symmetric $(0,2)$-tensors on $\mathbb{R}^3$. Suppose further that σ, τ are compactly supported on $\{1 \le |x| \le 2\}$ satisfying the linearized constraint equations (at the trivial data)

$$\sum_{i,j} (\sigma_{ij,ij} - \sigma_{ii,jj}) = 0,$$

and for $j = 1, 2, 3$,

$$\sum_i \tau_{ij,i} = 0.$$

Consider

$$\hat{g}_\lambda = g + \sigma_\lambda \qquad \text{and} \qquad \hat{\pi}_\lambda = \pi + \tau_\lambda,$$

where $\sigma_\lambda = \sigma(x/\lambda), \tau_\lambda = \tau(x/\lambda)$. To specify the center of mass and angular momentum, the proof centers on constructing the tensors σ, τ with certain desired properties. For the angular momentum, we need to construct σ, τ whose components satisfy $\sigma_{ij}(x) = \sigma_{ij}(-x)$, $\tau_{ij}(x) = \tau_{ij}(-x)$ such that for a given $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$,

$$\int_{\{1 \le |x| \le 2\}} \sum_{i,j,l} \left[\frac{1}{2} \tau_{ij,l} Y^l_{(k)} + \tau_{il} (Y^l_{(k)})_{,j} \right] \sigma^{ij} \, dx = \alpha_k$$

for each $k = 1, 2, 3$, where $Y_{(k)}$ is the rotation vector field $Y_{(k)} = \frac{\partial}{\partial x^k} \times \vec{x}$. To specify the center of mass, we need to construct a divergence-free and trace-free tensor σ such that for a given $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$,

$$\int_{\{1\le|x|\le 2\}} x^k \sum_{i,j,l} (\sigma_{ij,l})^2 \, dx = \gamma_k,$$

for each $k = 1, 2, 3$. We refer the construction of those tensors to [31, Theorem 2.1, Theorem 2.2].

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