THE EQUALITY CASE OF THE PENROSE INEQUALITY FOR ASYMPTOTICALLY FLAT GRAPHS

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Abstract. We prove the equality case of the Penrose inequality in all dimensions for asymptotically flat hypersurfaces. It was recently proven by G. Lam that the Penrose inequality holds for asymptotically flat graphical hypersurfaces in Euclidean space with non-negative scalar curvature and with a minimal boundary. Our main theorem states that if the equality holds, then the hypersurface is a Schwarzschild solution. As part of our proof, we show that asymptotically flat graphical hypersurfaces with a minimal boundary and non-negative scalar curvature must be mean convex, using the argument that we developed in [12]. This enables us to obtain the ellipticity for the linearized scalar curvature operator and to establish the strong maximum principles for the scalar curvature equation.

1. Introduction

The Penrose inequality in general relativity states that the ADM mass of an asymptotically flat manifold is at least the mass of the black holes that it contains, if the energy density is non-negative everywhere. A particularly important special case of the physical statement is called the Riemannian Penrose inequality.

The Riemannian Penrose Inequality Conjecture. *Let $(M^n, g), n \geq 3$, be an asymptotically flat n-dimensional smooth manifold with a strictly outer-minimizing smooth minimal boundary which is compact (not necessarily connected) of total $(n-1)$-volume A. Suppose that M has non-negative scalar curvature and the ADM mass m. Then*

$$m \geq \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$

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where ω_{n-1} is the volume of the unit $(n-1)$-sphere in Euclidean space. Moreover, the equality holds if and only if (M, g) is isometric to the region of a Schwarzschild metric outside its minimal hypersurface.

G. Huisken and T. Ilmanen proved the conjecture in dimension three for a connected minimal boundary [14]. H. Bray used a different approach and proved the conjecture in dimension three for any number of components of the minimal boundary [2]. In dimensions less than 8, the inequality was proved by H. Bray and D. Lee, with the extra spin assumption for the equality case [5]. In the case that (M, g) is conformally flat, H. Bray and K. Iga derived new property of superharmonic functions in $\mathbb{R}^n$ and proved the Penrose inequality with a suboptimal constant for $n = 3$ [4], F. Schwartz obtained a lower bound of the ADM mass in terms of the Euclidean volume of the region enclosed by the minimal boundary [19], and J. Jauregui proved a Penrose-like inequality [15]. For the Penrose inequality (with the sharp constant) in dimensions higher than 8, the only result that we know, other than the spherically symmetric case, is the result of G. Lam [16], where he proved that the Penrose *inequality* for graphical asymptotically flat hypersurfaces. (Note a related work regarding the Penrose inequality for asymptotically hyperbolic graphs [7, 8].)

Theorem 1 ([16])**.** *Let Ω be an open and bounded subset (not necessarily connected) in $\mathbb{R}^n$, $n \geq 3$, and let $f \in C^2(\mathbb{R}^n \setminus \overline{\Omega}) \cap C^0(\mathbb{R}^n \setminus \Omega)$ be asymptotically flat. We assume that the graph of f is a C^2 hypersurface up to boundary with non-negative scalar curvature. Suppose that each connected component of Ω is star-shaped*[1] *and each connected component of $\partial\Omega$ is the level set of f with $\langle Df(x), \eta(x)\rangle \to +\infty$ as $x \to \partial\Omega$, where η is the outward unit normal to the level sets of f. Then,*

$$m \geq \frac{1}{2}\left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$

where $|\partial\Omega|$ is the $(n-1)$-total volume of $\partial\Omega$.

The proof is simple and elegant, which we include in Section 5. However, the equality case was not discussed in [16], and the techniques there seem far from sufficient to handle the equality case. Our main result in this article proves the equality case in all dimensions $n \geq 3$. This may be particularly interesting because there was no rigidity statement for the Penrose inequality, other than the spherically symmetric case, known to hold for $n \geq 8$.

[1]In [16], each connected component of Ω was assumed convex, but the proof can be generalized to star-shaped domains.

Theorem 2. *Let Ω be an open and bounded subset (not necessarily connected) in $\mathbb{R}^n$, $n \geq 3$. Let $f \in C^{n+1}(\mathbb{R}^n \setminus \overline{\Omega}) \cap C^0(\mathbb{R}^n \setminus \Omega)$ be asymptotically flat. We assume that the graph of f is a C^2 hypersurface up to boundary with non-negative scalar curvature. Suppose that each connected component of Ω is star-shaped and each component of $\partial\Omega$ is the level set of f with $\langle Df(x), \eta(x)\rangle \to +\infty$ as $x \to \partial\Omega$, where η is the outward unit normal to the level sets of f. For $n = 3$ or 4, we assume additionally that $\max_{|x|=r} f(x) \leq \min_{|x|=r} f(x) + C$ for all r sufficiently large and for some constant C independent of r. If*

$$m = \frac{1}{2}\left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$

then the graph of f is identical to the region of the Schwarzschild solution of mass m outside its minimal $(n-1)$-hypersurface.

Remark 1.1. In dimensions less than 8, the above theorem is implied by more general results in [14, 3, 5] because hypersurfaces in Euclidean space are spin. Our proof is different and works for all dimensions. The additional assumption for $n = 3$ or 4 that

$$\max_{|x|=r} f(x) \leq \min_{|x|=r} f(x) + C \quad \text{for all } r \text{ sufficiently large}$$

means that the oscillation of f at infinity is controllable. We actually conjecture a stronger statement that an n-dimensional asymptotically flat hypersurface with zero scalar curvature has the following expansion at infinity

$$f(x) = \begin{cases} C_0\sqrt{|x|} + C_1 + o(1) & \text{if } n = 3 \\ C_0 \ln|x| + C_1 + o(1) & \text{if } n = 4, \end{cases} \tag{1.1}$$

for some constants C_0, C_1. This conjecture should compare with the celebrated work of R. Schoen on the uniqueness of catenoids [18], in which a preliminary result says that complete minimal hypersurfaces have specific asymptotics at infinity, up to lower order terms. In general, hypersurfaces with zero scalar curvature are more difficult to analyze than minimal hypersurfaces, because the scalar curvature equation of the graphing function is fully nonlinear. Assuming certain asymptotic behavior of the hypersurfaces at infinity (in all dimensions, which is stronger than (1.1) in the low dimensions) and the strict ellipticity condition, J. Hounie and M. Leite proved the uniqueness of embedded scalar-flat hypersurfaces with two ends [11].

Our proof of Theorem 2 relies on a key observation that an asymptotically flat graphical hypersurface with a minimal boundary and with non-negative scalar curvature must be mean convex. It is inspired by our earlier work [12], in which we proved that *closed* or certain *complete* hypersurfaces with non-negative scalar curvature must be mean convex.

Theorem 3. *Let Ω be an open and bounded subset (not necessarily connected) in $\mathbb{R}^n$. Let $f \in C^{n+1}(\mathbb{R}^n \setminus \overline{\Omega}) \cap C^0(\mathbb{R}^n \setminus \Omega)$ be asymptotically flat. Suppose that each connected component of $\partial\Omega$ is the level set of f with $\langle Df(x), \eta(x)\rangle \to +\infty$ as $x \to \partial\Omega$, where η is the outward unit normal to the level sets of f. If the scalar curvature of the graph of f is non-negative, then its mean curvature H has a sign, i.e., either $H \geq 0$ or $H \leq 0$ everywhere.*

The mean convexity enables us to derive the maximum principles for the scalar curvature equation. The proof of Theorem 2 is more delicate in the case $n = 3$ or 4, because the graphing function of the Schwarzschild solution tends to infinity as $|x| \to \infty$, and it is subtle to compare two unbounded graphs. To control the asymptotical behavior of f, we use its asymptotic flatness and develop a global strong maximum principle (Theorem 4.6) in the region where $|x|$ is sufficiently large. The maximum principles for the scalar curvature equation are established in Section 4.

Note that in our earlier work [12], we proved the Positive Mass Theorem for hypersurfaces in Euclidean space in all dimensions, including the rigidity statement, which is a direct consequence of our proof of the positive mass inequality. However, the proof of the equality case of the Penrose inequality requires a new argument, which is entirely different from our proof of the equality case of the Positive Mass Theorem.

In response to an interesting question raised by Christina Sormani and Dan Lee about the hypersurface which is a Schwarzschild solution outside a compact set, we have the following result.

Theorem 4. *There is no complete C^{n+1} hypersurface of one end with zero scalar curvature in $\mathbb{R}^{n+1}$ which is identical to a Schwarzschild solution with $m > 0$ outside a compact set.*[2]

The above theorem is in contrast to a general result of J. Corvino [6], where he constructed the complete asymptotically flat manifold with zero scalar curvature which is a Schwarzschild metric outside a compact set, but not identical to a Schwarzschild solution everywhere.

[2]Note that a Schwarzschild solution may not be uniquely embedded in Euclidean space as a hypersurface. Here, that a hypersurface is identical to a Schwarzschild solution outside a compact set is in the sense that the hypersurface is the graph of h outside a compact set of a hyperplane, where h is the radially symmetric function that gives a Schwarzschild solution in Proposition 2.6.

After this article was written and has been distributed among a small mathematics community, we noticed a preprint by L. de Lima and F. Girão [9] that announces a proof the rigidity case of the Penrose inequality using [11] and assuming ellipticity. Our proof is different, and the main part of our proof is to *derive* ellipticity and the maximum principles. We believe that our arguments will have future applications to hypersurfaces in space forms with the appropriate scalar curvature condition.

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2. Definitions, notation, and preliminary results

Definition 2.1. *Let Ω be a bounded subset in $\mathbb{R}^n$, $n \geq 3$. We say that $f \in C^2(\mathbb{R}^n \setminus \Omega)$ is asymptotically flat if the following conditions hold*

(1) $\lim_{|x|\to\infty} f(x) = C$ *for some bounded constant* C, $\lim_{|x|\to\infty} f(x) = \infty$, *or* $\lim_{|x|\to\infty} f(x) = -\infty$;

(2) $|Df(x)| = O(|x|^{-\frac{q}{2}})$ *and* $|D^2 f(x)| = O(|x|^{-\frac{q}{2}-1})$, *for some* $q > \frac{n-2}{2}$, *where* $Df = (f_1, \dots, f_n), D^2 f = (f_{ij})$ *and* $f_i = \partial f / \partial x^i$, $f_{ij} = \partial^2 f / \partial x^i \partial x^j$;

(3) *The scalar curvature of the graph of f is integrable over the graph of f.*

Remark 2.2. Under Condition (2), the induced metric of the graph of f has the asymptotics

$$g_{ij} = \delta_{ij} + f_i f_j = \delta_{ij} + O(|x|^{-q}).$$

The decay rate q is optimal in order for the ADM mass to be well-defined, assuming Condition (3) (see [1]).

Remark 2.3. Condition (1) in Definition 2.1 is not needed in the proof of Theorem 1. Condition (1) is actually redundant for $n \geq 6$ because by Condition (2) and using the mean value theorem along the radial direction and along the spherical direction, we have

$$\lim_{|x|\to\infty} f(x) = C,$$

for some bounded constant C.

Definition 2.4 ([12, 16])**.** *Let Ω be a bounded subset in $\mathbb{R}^n$, $n \geq 3$, and let $f \in C^2(\mathbb{R}^n \setminus \Omega)$. The mass of the graph of f is defined by*

$$m = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \frac{1}{1+|Df|^2} \sum_{i,j} (f_{ii} f_j - f_{ij} f_i) \frac{x^j}{|x|} \, d\sigma,$$

where $S_r = \{(x^1, \ldots, x^n) : |x| = r\}$*, and* $d\sigma$ *is the standard spherical volume measure of* S_r*.*

Remark 2.5. The above definition of the mass is consistent with the classical definition of the ADM mass [12, Lemma 5.8], *cf.* [16].

The spacelike n-dimensional Schwarzschild metric is a complete and conformally flat metric on $\mathbb{R}^n \setminus \{0\}$

$$\left(\mathbb{R}^n \setminus \{0\}, \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta\right),$$

where m is the ADM mass. If $m \geq 0$, the n-dimensional Schwarzschild solution can be isometrically embedded into Euclidean $\mathbb{R}^{n+1}$ as a smooth hypersurface. We refer the reader to [3] for detailed discussions, especially for the $n = 3$ case. We are interested in the region of the Schwarzschild solution outside its minimal $(n-1)$-hypersurface, which is graphical as shown in the following proposition.

Proposition 2.6. *Denote by* B_r *the open ball in* $\mathbb{R}^n$ *centered at the origin of radius* r*. The region of the Schwarzschild solution of mass* $m > 0$ *outside its minimal* $(n-1)$*-hypersurface can be represented as the graph of* $h(x)$ *over* $\mathbb{R}^n \setminus B_{(2m)^{1/(n-2)}}$*, where*

$$\begin{aligned}
&h(x) = C_0 + \sqrt{8m(|x| - 2m)} && \text{if } n = 3,\\
&h(x) = C_0 + \sqrt{2m}\ln(|x| + \sqrt{|x|^2 - 2m}) && \text{if } n = 4,\\
&h(x) = C_0 + O(|x|^{2-\frac{n}{2}}) \quad \text{for } |x| \gg 1 && \text{if } n \geq 5,
\end{aligned}$$

for some constant C_0*.*

Proof. Let $h \in C^2(\mathbb{R}^n \setminus B_{r_0})$ for some $r_0 \geq 0$ be rotationally symmetric. With a minor abuse of notation, we will write $h(x) = h(r)$ where $r = |x|$. By direct computation, the scalar curvature R of the graph of h is given by

$$\frac{R}{2} = \frac{(n-1)h''h'}{r\left[1 + (h')^2\right]^2} + \frac{\binom{n-1}{2}(h')^2}{r^2\left[1 + (h')^2\right]},$$

where $h' = \frac{dh}{dr}$ and $h'' = \frac{d^2h}{dr^2}$. Set $y(r) = -\frac{1}{1+(h')^2}$. Then $-1 \leq y \leq 0$ and y solves

$$y' + \frac{n-2}{r}y + \frac{n-2}{r} = \frac{rR}{2(n-1)}.$$

If $R \equiv 0$, for some constant $C_1 \geq 0$, we have

$$y = C_1 r^{2-n} - 1.$$

Therefore, for $r > (C_1)^{1/(n-2)}$,

$$(h')^2 = \frac{1}{1 - C_1 r^{2-n}} - 1 = \frac{C_1 r^{2-n}}{1 - C_1 r^{2-n}} = \frac{C_1}{r^{n-2} - C_1}.$$

Then,

$$h(r) = \sqrt{C_1} \int \frac{1}{\sqrt{r^{n-2} - C_1}}\, dr.$$

Solving the integral, we have, for some constant C_0,

$$\begin{aligned} h(r) &= C_0 + \sqrt{4C_1(r - C_1)} && \text{if } n = 3,\\ h(r) &= C_0 + \sqrt{C_1} \ln(r + \sqrt{r^2 - C_1}) && \text{if } n = 4,\\ h(r) &= C_0 + O(r^{2-\frac{n}{2}}) \quad \text{for } r \gg 1 && \text{if } n \geq 5. \end{aligned}$$

By computing the mass directly, we have

$$m = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \frac{(n-1)(h')^2}{r(1 + (h')^2)}\, d\sigma = \frac{C_1}{2}.$$

It is straightforward to check that if $m > 0$, $h'(r) \to \infty$ as $r \to (2m)^{1/(n-2)}$, and the graph of h over $\partial B_{(2m)^{1/(n-2)}}$ is the minimal $(n-1)$-hypersurface in the graph of h. $\square$

Notation. For a hypersurface, we denote by A_{ij} the second fundamental form, by $A^i_j = \sum_k g^{ik} A_{kj}$ the shape operator where g^{ik} is the inverse of the induced metric, by H the mean curvature, and by R the scalar curvature. If the hypersurface is the graph of u, we compute A_{ij} with respect to the *upward* unit normal vector and we can write the above quantities as the functions of Du and D^2u. We may also suppress the arguments when the context is clear.

$$\begin{aligned} g^{ik}(Du) &= \left(\delta_{ik} - \frac{u_i u_k}{1 + |Du|^2}\right)\\ A_{ij}(Du, D^2u) &= \frac{u_{ij}}{\sqrt{1 + |Du|^2}}\\ A^i_j(Du, D^2u) &= \sum_k \left(\delta_{ik} - \frac{u_i u_k}{1 + |Du|^2}\right) \frac{u_{kj}}{\sqrt{1 + |Du|^2}}\\ H(Du, D^2u) &= \sum_{i,j} \left(\delta_{ij} - \frac{u_i u_j}{1 + |Du|^2}\right) \frac{u_{ij}}{\sqrt{1 + |Du|^2}}\\ R(Du, D^2u) &= H^2(Du, D^2u) - \sum_{i,j} A^i_j(Du, D^2u) A^j_i(Du, D^2u). \end{aligned}$$

Proposition 2.7. *Let h be the graphing function of the Schwarzschild solution in Proposition 2.6. Then the matrix $\left(Hg^{ij} - \sum_k A^i_k g^{kj}\right)$ of the graph of h is positive definite everywhere in $\mathbb{R}^n \setminus B_{(2m)^{1/(n-2)}}$.*

Proof. It suffices to show that $(H\delta^i_k - A^i_k)$ is positive definite, because $Hg^{ij} - \sum_k A^i_k g^{kj} = \sum_k (H\delta^i_k - A^i_k) g^{kj}$ and (g^{kj}) is positive definite. By rotating coordinates, we can assume that $A^i_k = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where λ_l are the principle curvatures. For the graph of a rotationally symmetric function $h(r)$, the principle curvatures are

$$\frac{h''}{(1+(h')^2)^{3/2}}, \quad \text{and} \quad \frac{h'}{r\sqrt{1+(h')^2}} \text{ with multiplicity } (n-1).$$

Therefore, the principle curvatures of the Schwarzschild solution are

$$-\frac{n-2}{2}\sqrt{2m}r^{-\frac{n}{2}}, \quad \text{and} \quad \sqrt{2m}r^{-\frac{n}{2}} \text{ with multiplicity } (n-1).$$

Hence, $(H\delta^i_k - A^i_k) \geq \frac{n-2}{2}\sqrt{2m}r^{-\frac{n}{2}} I$ where I is the $n\times n$ identity matrix, so it is positive definite everywhere in $\mathbb{R}^n \setminus B_{(2m)^{1/(n-2)}}$. □

The following two propositions were proven in our earlier paper [12]. They play important roles to prove the mean convexity of the asymptotically flat graphs and to derive the ellipticity of the linearized scalar curvature operator.

Proposition 2.8 ([12, Proposition 2.1])**.** *Let $B = (b_{ij})$ be an $n \times n$ matrix with $n \geq 2$. Denote*

$$\begin{aligned} \sigma_1(B) &= \sum_{i=1}^n b_{ii}, \quad \sigma_1(B|k) = \left(\sum_{i=1}^n b_{ii}\right) - b_{kk}, \\ \sigma_2(B) &= \sum_{1\leq i<j\leq n} (b_{ii}b_{jj} - b_{ij}b_{ji}). \end{aligned}$$

For each $1 \leq k \leq n$, we have

$$\begin{aligned} \sigma_1(B)\sigma_1(B|k) = \sigma_2(B) &+ \frac{n}{2(n-1)}\left(\sigma_1(B|k)\right)^2 + \sum_{1\leq i<j\leq n} b_{ij}b_{ji} \\ &+ \frac{1}{2(n-1)} \sum_{\substack{1\leq i<j\leq n \\ i\neq k, j\neq k}} (b_{ii} - b_{jj})^2, \end{aligned}$$

where the last term is zero when $n = 2$. In particular, if B is real and $b_{ij}b_{ji} \geq 0$ for all $1 \leq i < j \leq n$, then

$$\sigma_1(B)\sigma_1(B|k) \geq \sigma_2(B) + \frac{n}{2(n-1)}\left(\sigma_1(B|k)\right)^2$$

with equality if and only if b_{ii} are equal for all $i = 1, \dots, n$ and $i \neq k$, and $b_{ij}b_{ji} = 0$ for all $i, j = 1, \dots, n$ and $i \neq j$.

Notation. Let N be a (piece of) hypersurface in Euclidean space, and let μ be a unit normal vector field to N. The mean curvature of N defined by μ is given by

$$H_N = -\mathrm{div}_0 \mu,$$

where div_0 is the Euclidean divergence operator. (The n-dimensional sphere of radius r has mean curvature n/r with respect to the inward unit normal vector by this convention.) We denote by $\langle \cdot, \cdot \rangle$ the standard metric on Euclidean space. With a slight abuse of notation, we may view η as a vector in $\mathbb{R}^n$, as well as a vector in $\mathbb{R}^{n+1}$ by letting the last component be zero.

Proposition 2.9 ([12, Theorem 2.2]). *Let M be a C^2 hypersurface in $\mathbb{R}^{n+1}$. Consider the height function $h : M \to \mathbb{R}$ given by $h(x^1, \dots, x^{n+1}) = x^{n+1}$. Let a be a regular value of h. Denote by*

$$\Sigma = M \cap \{x^{n+1} = a\},$$

which is a C^2 hypersurface in $\{x^{n+1} = a\}$ and $|\nabla^M h| > 0$ at every point in Σ. Denote by ν and η the unit normal vector fields to $M \subset \mathbb{R}^{n+1}$ and $\Sigma \subset \{x^{n+1} = a\}$, respectively, and denote by H and H_Σ the mean curvatures of $M \subset \mathbb{R}^{n+1}$ and $\Sigma \subset \{x^{n+1} = a\}$ defined by ν and η, respectively. Let R be the induced scalar curvature of M. Then, at every point of Σ,

$$\langle \nu, \eta \rangle H H_\Sigma \geq \frac{R}{2} + \frac{n}{2(n-1)} \langle \nu, \eta \rangle^2 H_\Sigma^2$$

with the equality at a point in Σ if and only if (M, Σ) satisfies the following two conditions at the point:

(i) *$\Sigma \subset \mathbb{R}^n$ is umbilic, with the principal curvature κ;*
(ii) *$M \subset \mathbb{R}^{n+1}$ has at most two distinct principal curvatures, and one of them is equal to $\langle \nu, \eta \rangle \kappa$, with multiplicity at least $n - 1$.*

3. Proof of Theorem 3

Notation. Let M be a hypersurface in Euclidean space and let $\mathrm{int}(M)$ be the set of *interior* points in M, i.e., $\mathrm{int}(M) = M \setminus \partial M$. The set of *interior geodesic points* is given by

$$M_0 = \{p \in \mathrm{int}(M) : (A^i_j) = 0 \text{ at } p\}. \tag{3.1}$$

A classical result of R. Sacksteder [17, Lemma 6] characterizes the set of geodesic points. Although he proved the statement for complete hypersurfaces, the statement can be easily generalized to hypersurfaces with boundary, see also [12, Lemma 3.8] and [13, Lemma 4.5]. In particular, in the later reference we showed the analogous statement for hypersurfaces in $\mathbb{S}^{n+1}$ with boundary.

Lemma 3.1. *Let M be a C^{n+1} hypersurface in $\mathbb{R}^{n+1}$, and let M_0' be a connected component of M_0. Then M_0' lies in a hyperplane which is tangent to M at every point in M_0'.*

To prove Theorem 3, let us recall the following results in [12].

Definition 3.2. *Let W be a subset in $\mathbb{R}^n$. A point $p \in \partial W$ is called a convex point of W, if there exists an $(n-1)$-sphere S in $\mathbb{R}^n$ passing through p so that $\overline{W} \setminus \{p\}$ is contained in the open ball enclosed by S. Here $\overline{W}$ denotes the closure of W in $\mathbb{R}^n$. Note that a bounded subset in $\mathbb{R}^n$ has at least one convex point.*

Lemma 3.3 ([12, Lemma 3.5])**.** *Let W be an open subset in $\mathbb{R}^n$, not necessarily bounded. Suppose that $p \in \partial W$ is a convex point of W. Denote by $B(p)$ an open ball in $\mathbb{R}^n$ centered at p. Suppose that $u \in C^2(W \cap B(p)) \cap C^1(\overline{W} \cap B(p))$ and $u = C, |Du| = 0$ on $\partial W \cap B(p)$ for some constant C. If the scalar curvature of the graph of u is non-negative, then the mean curvature of the graph of u must change signs in $W \cap B(p)$, unless $u \equiv C$ on $W \cap B(p)$.*

Lemma 3.4 ([12, Proposition 3.1])**.** *Let W be an open subset in $\mathbb{R}^n$, not necessarily bounded. Let $p \in \partial W$, and denote by $B(p)$ a small open ball in $\mathbb{R}^n$ centered at p. Suppose that $f \in C^2(W \cap B(p)) \cap C^1(\overline{W} \cap B(p))$ satisfies*

$$\begin{aligned} &H(Df, D^2 f) \geq 0 && \text{in } W \cap B(p) \\ &f = C,\ |Df| = 0 && \text{on } \partial W \cap B(p), \end{aligned}$$

for some constant C. Then either $f \equiv C$ in $W \cap B(p)$, or

$$\{x \in W \cap B(p) : f(x) > C\} \neq \emptyset.$$

Definition 3.5. *Let N be an open manifold and N_1 and N_2 be two nonempty disconnected open subsets of N. For a closed subset E in N, we say that E separates N_1 and N_2 in N, if there exists an open neighborhood V of E in N so that $V \setminus E = N_1 \cup N_2$.*

Proof of Theorem 3. Denote by M the graph of f. First notice that by the Gauss equation the condition $R \geq 0$ implies that the set of geodesic points M_0 equals $\{p \in \mathrm{int}(M) : H = 0 \text{ at } p\}$. Suppose on the contrary that the mean curvature changes signs. Then, there exists a connected component of M_0, say M_0', which separates the regions $\{p \in \operatorname{int}(M) : H > 0 \text{ at } p\}$ and $\{p \in \operatorname{int}(M) : H < 0 \text{ at } p\}$ in M. By Lemma 3.1, M_0' lies in a hyperplane Π tangent to M at M_0'. $\overline{M_0'}$ does not intersect with ∂M because $|Df| = \infty$. M can be represented as the graph of a C^{n+1}-function u in an open neighborhood of M_0' in Π, and u satisfies $u = 0, |Du| = 0$ on M_0'.

We first **claim** that M_0' cannot be bounded. Suppose on the contrary that M_0' is bounded. Because M_0' separates $\{p \in \operatorname{int}(M) : H > 0 \text{ at } p\}$ and $\{p \in \operatorname{int}(M) : H < 0 \text{ at } p\}$ in M and locally M is the graph of u over a neighborhood of M_0' in Π, M_0' also separates the regions in Π. Let W be the open subset of Π enclosed by M_0'. Then $\partial W \subset M_0'$. Note that W is bounded and hence has a convex point $p \in \partial W$. For a small open ball $B(p)$, we have either $H > 0$, or $H < 0$ everywhere in $W \cap B(p)$. However, it contradicts Lemma 3.3. We prove the claim.

Thus, M_0' is unbounded. By the assumption that f is asymptotically flat, the upward unit normal vector of M converges to ∂_{n+1} at infinity. Because M is tangent to the hyperplane Π at an unbounded set, we must have $\lim_{|x|\to\infty} f(x) = C$ for some bounded constant C and $\Pi = \{x^{n+1} = C\}$. Hence $u = f - C$ in a neighborhood of M_0' in Π. Let ν be the upward unit normal vector field on M

$$\nu = \frac{(-Df, 1)}{\sqrt{1 + |Df|^2}} \quad \text{at } (x, f(x)) \in M.$$

Because M_0' separates $\{p \in \operatorname{int}(M) : H > 0 \text{ at } p\}$ and $\{p \in \operatorname{int}(M) : H < 0 \text{ at } p\}$ in M, the set $\{p \in \operatorname{int}(M) : H > 0 \text{ at } p\}$ is non-empty near M_0'. By Lemma 3.4, the level set $\{x : f(x) = C+\epsilon\}$ has non-empty intersection with $\{p \in \operatorname{int}(M) : H > 0 \text{ at } p\}$ for all $\epsilon > 0$ sufficiently small. Let $\Sigma_{C+\epsilon}$ be a connected component of the level set which has non-empty intersection with $\{p \in \operatorname{int}(M) : H > 0 \text{ at } p\}$. Note that $\Sigma_{C+\epsilon}$ is closed if $\epsilon \neq 0$, and that $H \geq 0$ at every point of $\Sigma_{C+\epsilon}$ because by the claim H can only change signs through an unbounded subset of M_0, which must lie on $\{x^{n+1} = C\}$. By Morse-Sard theorem, $\Sigma_{C+\epsilon}$ is a C^{n+1} submanifold with $|Df| > 0$ for almost every ϵ. Let $\eta = Df/|Df|$ be a unit normal vector of $\Sigma_{C+\epsilon}$ where $C + \epsilon$ is a regular value. For $\epsilon > 0$ sufficiently small, η points *inward* to the region enclosed by $\Sigma_{C+\epsilon}$ because f decreases to C at infinity. Let $H_{\Sigma_{C+\epsilon}}$ be the mean curvature with respect to η. By Proposition 2.9, $H_{\Sigma_{C+\epsilon}} \leq 0$ at every point of $\Sigma_{C+\epsilon}$, which contradicts compactness of $\Sigma_{C+\epsilon}$. $\square$

4. Ellipticity and maximum principles

In this section, we will derive various maximum principles for the scalar curvature equation. The scalar curvature equation of the graphing function is fully nonlinear. Its linearization gives a second-order differential equation, which, however, may not be elliptic in general. An important step to establish the maximum principles is to explore the (strict) ellipticity of the linearized scalar curvature equation.

Lemma 4.1. *Let u be a C^2 function and let R be the scalar curvature of the graph of u. Then*

$$\frac{\partial R}{\partial u_{ij}}(Du, D^2u) = \frac{2}{\sqrt{1+|Du|^2}}\left(Hg^{ij} - \sum_k A^i_k g^{kj}\right).$$

Proof. By chain rule,

$$\begin{aligned}
\frac{\partial R}{\partial u_{ij}} &= \sum_{k,l} \frac{\partial R}{\partial A^k_l}\frac{\partial A^k_l}{\partial u_{ij}} \\
&= \sum_{k,l} \frac{\partial R}{\partial A^k_l}\frac{\partial}{\partial u_{ij}}\left(\sum_p g^{kp}A_{pl}\right) \\
&= \sum_k \frac{\partial R}{\partial A^k_i}\frac{g^{kj}}{\sqrt{1+|Du|^2}} \\
&= 2H\frac{g^{ij}}{\sqrt{1+|Du|^2}} - 2\sum_k A^i_k \frac{g^{kj}}{\sqrt{1+|Du|^2}}.
\end{aligned}$$

□

Proposition 4.2. *Let u be a C^2 function, and let R and H be the scalar curvature and mean curvature of the graph of u, respectively. If $R \geq 0$ and $H \geq 0$, then the matrix $\left(Hg^{ij} - \sum_k A^i_k g^{kj}\right)$ is semi-positive definite.*

Proof. Because $Hg^{ij} - \sum_k A^i_k g^{kj} = \sum_k (H\delta^i_k - A^i_k)g^{kj}$ and (g^{kj}) is positive definite, it suffices to prove that $(H\delta^i_k - A^i_k)$ is semi-positive definite. By rotating the coordinates, we assume $(A^i_k) = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Then

$$(H\delta^i_k - A^i_k) = \mathrm{diag}(\sigma_1(A|1), \dots, \sigma_n(A|n)).$$

By Proposition 2.8, because $H = \sigma_1(A) \geq 0$ and $R = 2\sigma_2(A) \geq 0$, we have $\sigma_1(A|k) \geq 0$ for all $k = 1, \dots, n$. □

Theorem 4.3 (Strong maximum principle for the interior point)**.** *Let Ω be a connected open subset in $\mathbb{R}^n$. Suppose $u, v \in C^2(\Omega)$, $u \geq v$ in*

Ω, and u, v satisfy

$$R(Du, D^2u) = 0, \quad R(Dv, D^2v) \geq 0,$$
$$H(Du, D^2u) \geq 0, \quad \text{and} \quad H(Dv, D^2v) \geq 0 \quad \text{in } \Omega.$$

We assume that either u or v satisfies $\left(Hg^{ij} - \sum_k A^i_k g^{kj}\right)$ being positive definite in Ω. If $u = v$ at some point in Ω, then $u \equiv v$ in Ω.

Proof. Let $R(p, \xi) \in C^1(\mathbb{R}^n \times \mathbb{R}^{n \times n})$ be the scalar curvature operator.

$$\begin{aligned} 0 &\geq R(Du, D^2u) - R(Dv, D^2v) \\ &= R(Du, D^2u) - R(Du, D^2v) + R(Du, D^2v) - R(Dv, D^2v) \\ &= \sum_{i,j} a^{ij}(u_{ij} - v_{ij}) + \sum_i b^i\,(u_i - v_i). \end{aligned}$$

where

$$b^i = \int_0^1 \frac{\partial R}{\partial p_i}(tDu + (1-t)Dv, D^2v)\, dt$$

and by Lemma 4.1

$$\begin{aligned} a^{ij} &= \int_0^1 \frac{\partial R}{\partial \xi_{ij}}(Du, tD^2u + (1-t)D^2v)\, dt \\ &= \frac{1}{\sqrt{1 + |Du|^2}}\Bigg(\Big(H(Du, D^2u)g^{ij}(Du) - \sum_k A^i_k(Du, D^2u)g^{kj}(Du)\Big) \\ &\quad + \Big(H(Du, D^2v)g^{ij}(Du) - \sum_k A^i_k(Du, D^2v))g^{kj}(Du)\Big)\Bigg). \end{aligned}$$

If $u = v$ at $p \in \Omega$, then $Du = Dv$ at p and (a^{ij}) is positive definite at p, by assumption and Proposition 4.2. By continuity, (a^{ij}) is positive definite in an open neighborhood Ω' of p in Ω. Then by the standard strong maximum principle, $u \equiv v$ in Ω'. Hence, the set $\{p \in \Omega : u(p) = v(p)\}$ is open and closed. Because Ω is connected, we prove that $u \equiv v$ in Ω. $\square$

Theorem 4.4 (Strong maximum principle for the boundary point)**.** *Let Ω_1, Ω_2 be connected open sets in $\mathbb{R}^n$ such that $\Omega_1 \subset \Omega_2$. Suppose that $p \in \partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$ and $\partial\Omega_1, \partial\Omega_2$ are C^1 near p.*

Let $u \in C^2(\Omega_1) \cap C^0(\overline{\Omega}_1), v \in C^2(\Omega_2) \cap C^0(\overline{\Omega}_2)$. Suppose that the graphs of u, v are C^2 hypersurfaces up to boundary which satisfy

$$R(Du, D^2u) = 0, \quad R(Dv, D^2v) \geq 0$$
$$H(Du, D^2u) \geq 0, \quad \text{and} \quad H(Dv, D^2v) \geq 0 \quad \text{in } \Omega_1.$$

We also assume that either u or v satisfies $\left(Hg^{ij} - \sum_k A^i_k g^{kj}\right)$ being positive definite in Ω_1. If $u \geq v \geq 0$ in Ω_1 and $u|_{\partial\Omega_1 \cap B_r(p)} =$

$v|_{\partial\Omega_2 \cap B_r(p)} = 0$ *for some open ball centered at* p *of radius* r *with* $|Du(x)|$, $|Dv(x)| \to \infty$ *as* $x \to p \in \partial\Omega_1 \cap \partial\Omega_2$*, then* $u \equiv v$ *in* Ω_1*.*

Proof. Let Π be the vertical hyperplane in $\mathbb{R}^{n+1}$ so that Π is tangent to $\partial\Omega_1 \times \{x^{n+1}\text{-axis}\}$ at $p \times \{x^{n+1}\text{-axis}\}$. The graphs of u, v near p can be locally represented as the graphs of some functions $\tilde{u}, \tilde{v}$ over a subset of Π, say $\tilde{u}, \tilde{v} \in C^2(D \times [0, \epsilon])$ where $p \in \operatorname{int}(D) \subset \{x^{n+1} = 0\}$ and $D \times [0, \epsilon] \subset \Pi$. Moreover, $\tilde{u}$ and $\tilde{v}$ satisfy $\tilde{u} \geq \tilde{v}$ in $D \times [0, \epsilon]$, $\tilde{u} = \tilde{v}$ and $\partial_{n+1}\tilde{u} = \partial_{n+1}\tilde{v} = 0$ at $p \in D \times \{0\}$, and

$$
\begin{aligned}
&R(D\tilde{u}, D^2\tilde{u}) = 0, \quad R(D\tilde{v}, D^2\tilde{v}) \geq 0\\
&H(D\tilde{u}, D^2\tilde{u}) \geq 0, \quad \text{and} \quad H(D\tilde{v}, D^2\tilde{v}) \geq 0 \quad \text{ in } D \times [0, \epsilon].
\end{aligned}
$$

Either $\tilde{u}$ or $\tilde{v}$ has $\left(Hg^{ij} - \sum_k A^i_k g^{kj}\right)$ being positive definite in $D \times [0, \epsilon]$. As analyzed in the proof of Theorem 4.3, $(\tilde{u} - \tilde{v})$ satisfies

$$0 \geq \sum_{i,j} a^{ij}(\tilde{u}_{ij} - \tilde{v}_{ij}) + \sum_i b^i(\tilde{u}_i - \tilde{v}_i),$$

where (a^{ij}) is positive definite in $D \times [0, \epsilon]$ with a possibly smaller ϵ. Then by the standard Hopf boundary point lemma, $\tilde{u} = \tilde{v}$ at some interior points of $D \times [0, \epsilon]$. Hence $u = v$ at some interior points in Ω_1, and by Theorem 4.3, $u \equiv v$ everywhere in Ω_1. $\square$

To prove Theorem 4, we need the following version of the strong maximum principle for the boundary point, where the domains of u and v are complement to each other. The proof is nearly identical to the proof of Theorem 4.4, so we omit it.

Theorem 4.5. *Let* Ω *be an open subset in* $\mathbb{R}^n$*. Let* $p \in \partial\Omega$ *and consider the open ball* $B_r(p)$ *centered at* p *of radius* r *for some* $r > 0$ *small. Suppose* $\partial\Omega \cap B_r(p)$ *is* C^1*. Let* $u \in C^2(B_r(p) \setminus \overline{(\Omega \cap B_r(p))}) \cap C^0(B_r(p) \setminus (\Omega \cap B_r(p)))$, $v \in C^2(\Omega \cap B_r(p)) \cap C^0(\overline{\Omega \cap B_r(p)})$ *and* $u \leq 0, v \leq 0$*. Suppose that the graphs of* u, v *are* C^2 *hypersurfaces up to boundary which satisfy*

$$
\begin{aligned}
&R(Du, D^2u) = 0, \quad R(Dv, D^2v) = 0\\
&H(Du, D^2u) \leq 0, \quad \textit{and} \quad H(Dv, D^2v) \geq 0.
\end{aligned}
$$

We also assume that the matrix $\left(Hg^{ij} - \sum_k A^i_k g^{kj}\right)$ *of* u *is negative definite. If* $u|_{\partial\Omega \cap B_r(p)} = v|_{\partial\Omega \cap B_r(p)} = 0$*. then* $|Du(x)|$ *and* $|Dv(x)|$ *cannot both tend to* ∞ *as* $x \to p \in \partial\Omega$*.*

Theorem 4.6 (Global strong maximum principle)**.** *Let* Ω *be a bounded subset (not necessarily connected) in* $\mathbb{R}^n$*, and let* $v \in C^2(\mathbb{R}^n \setminus \Omega)$ *be asymptotically flat. We assume that the graph of* v *satisfies* $R = 0$

and $H \geq 0$ in $\mathbb{R}^n \setminus \Omega$. Let h be the Schwarzschild solution given by Proposition 2.6. Then there exists $r \gg 1$ so that, for any $r_2 > r_1 \geq r$,

$$\max_{\overline{B}_{r_2} \setminus B_{r_1}} (h - v) = \max_{S_{r_2} \cup S_{r_1}} (h - v)$$
$$\min_{\overline{B}_{r_2} \setminus B_{r_1}} (h - v) = \min_{S_{r_2} \cup S_{r_1}} (h - v).$$

If $(h - v)$ attains its maximum or minimum at an interior point in $B_{r_2} \setminus \overline{B}_{r_1}$, then $(h - v)$ must be a constant in $\overline{B}_{r_2} \setminus B_{r_1}$.

Proof. As computed in the proof of Theorem 4.3, we have

$$\begin{aligned} 0 &= R(Dh, D^2h) - R(Dv, D^2v) \\ &= \sum_{i,j} a^{ij}(h_{ij} - v_{ij}) + \sum_i b^i\,(h_i - v_i). \end{aligned}$$

where

$$\begin{aligned} a^{ij} = \frac{1}{\sqrt{1 + |Dh|^2}} \sum_k \Big(& H(Dh, D^2h)\delta^i_k - A^i_k(Dh, D^2h) \\ & + H(Dh, D^2v)\delta^i_k - A^i_k(Dh, D^2v) \Big) g^{kj}(Dh). \end{aligned}$$

We shall prove that (a_{ij}) is positive definite in $\mathbb{R}^n \setminus B_r$ for $r \gg 1$. Then the lemma follows directly from the standard maximum principles. Because (g^{kj}) is positive definite, we can prove the positivity of (a^{ij}) by showing that the matrix

$$(H(Dh, D^2h)\delta^i_k - A^i_k(Dh, D^2h) + H(Dh, D^2v)\delta^i_k - A^i_k(Dh, D^2v)) \tag{4.1}$$

is positive definite. By direct computation,

$$\begin{aligned} & H(Dh, D^2v)\delta^i_k - A^i_k(Dh, D^2v) \\ &= H(Dv, D^2v)\delta^i_k - A^i_k(Dv, D^2v) + O(|Dv|^2|D^2v| + |Dh|^2|D^2h|) \\ &= H(Dv, D^2v)\delta^i_k - A^i_k(Dv, D^2v) + o\left(r^{(-3n+2)/4}\right), \end{aligned}$$

where $r = |x|$ and we use the asymptotic flatness of h and v. By Proposition 4.2, $(H(Dv, D^2v)\delta^i_k - A^i_k(Dv, D^2v))$ is semi-positive definite. By Proposition 2.7,

$$\big(H(Dh, D^2h)\delta^i_k - A^i_k(Dh, D^2h)\big) \geq \frac{n-2}{2}\sqrt{2m} r^{-n/2} I.$$

The right hand side above is positive enough to absorb the error term $o(r^{(-3n+2)/4})$ if $r \gg 1$. Hence (4.1) is positive definite. ∎

5. Proofs of Theorem 2 and Theorem 4

We need the the following elementary inequalities. The first inequality is a special case of the Alexandrov-Fenchel inequalities. The classical Alexandrov-Fenchel inequalities were proven for convex domains. It has been generalized to star-shaped domains [10].

Proposition 5.1 ([10])**.** *Let $\Omega \subset \mathbb{R}^n$ be star-shaped and let $\Sigma = \partial\Omega$. Denote by H_Σ the mean curvature of Σ with respect to the inward unit normal vector. Then*

$$\frac{1}{2(n-1)\omega_{n-1}} \int_\Sigma H_\Sigma \, d\sigma \geq \frac{1}{2} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$$

with equality if and only if Σ is an $(n-1)$-sphere.

Proposition 5.2. *Let $a_1, a_2, \ldots, a_k$ be non-negative real numbers and $0 \leq \beta \leq 1$. Then*

$$\sum_{i=1}^k a_i^\beta \geq \left(\sum_{i=1}^k a_i \right)^\beta.$$

If $0 \leq \beta < 1$, the equality holds if and only if at most one of a_i is non-zero.

Proof. Without loss of generality, we assume $k = 2$. We shall prove that $x^\beta + y^\beta \geq (x+y)^\beta$ if $0 \leq \beta \leq 1$ and $x, y \geq 0$. Fix β, x and define $w(y) = x^\beta + y^\beta - (x+y)^\beta$. Then $w(0) = 0$ and

$$w'(y) = \beta \left(y^{\beta-1} - (x+y)^{\beta-1} \right) \geq 0 \quad \text{for all } y \geq 0.$$

Hence $w(y) \geq 0$ for all $y \geq 0$ with $w(y) = 0$ if and only if $x = 0$, or $y = 0$, or $\beta = 1$. □

Proof of Theorem 1 ([16])**.** The scalar curvature of the graph of f has a divergence form (see also [12, Proposition 5.4])

$$R = \sum_j \partial_j \sum_i \left(\frac{f_{ii} f_j - f_{ij} f_i}{1 + |Df|^2} \right).$$

Let Ω^ϵ be a bounded open set in $\mathbb{R}^n$ that contains Ω with $\Omega^\epsilon \to \Omega$ as $\epsilon \to 0$, and let each connected component Σ_k^ϵ of $\partial\Omega^\epsilon$ be the level set of

f. By applying the divergence theorem over $\mathbb{R}^n \setminus \Omega^\epsilon$, we have

$$\begin{aligned}
&2(n-1)\omega_{n-1}m \\
&= \lim_{r\to\infty}\int_{S_r}\frac{1}{1+|Df|^2}\sum_{i,j}(f_{ii}f_j - f_{ij}f_i)\frac{x^j}{|x|}\,d\sigma \\
&= \int_{\mathbb{R}^n\setminus\Omega^\epsilon} R\,dx - \sum_k\int_{\Sigma_k^\epsilon}\frac{1}{1+|Df|^2}\sum_{i,j}(f_{ii}f_j - f_{ij}f_i)\eta^j\,d\sigma \\
&= \int_{\mathbb{R}^n\setminus\Omega^\epsilon} R\,dx + \sum_k\int_{\Sigma_k^\epsilon}\frac{|Df|^2}{1+|Df|^2}H_{\Sigma_k^\epsilon}\,d\sigma,
\end{aligned}$$

where $H_{\Sigma_k^\epsilon}$ denotes the mean curvature of the level set Σ_k^ϵ with respect to the unit normal vector η pointing *inward* to the region enclosed by Σ_k^ϵ (*cf.* [12, Proof of Lemma 5.6]). Let ϵ tend to zero. Then each level set Σ_k^ϵ tends to the connected component Σ_k of $\partial\Omega$ and $|Df| \to \infty$. Therefore, we have

$$\begin{aligned}
m &= \frac{1}{2(n-1)\omega_{n-1}}\left(\int_{\mathbb{R}^n\setminus\Omega} R\,dx + \sum_k\int_{\Sigma_k} H_{\Sigma_k}\,d\sigma\right) \\
&\ge \sum_k \frac{1}{2(n-1)\omega_{n-1}}\int_{\Sigma_k} H_{\Sigma_k}\,d\sigma \\
&\ge \sum_k \frac{1}{2}\left(\frac{|\Sigma_k|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} \qquad \text{(by Proposition 5.1)} \\
&\ge \frac{1}{2}\left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} \qquad \text{(by Proposition 5.2).}
\end{aligned}$$

□

Corollary 5.3. *Let Ω be an open and bounded subset (not necessarily connected) in $\mathbb{R}^n$, $n \ge 3$, and let $f \in C^2(\mathbb{R}^n \setminus \overline{\Omega}) \cap C^0(\mathbb{R}^n \setminus \Omega)$ be asymptotically flat. We assume that the graph of f is a C^2 hypersurface up to boundary with non-negative scalar curvature. Suppose that each connected component of Ω is star-shaped and each connected component of $\partial\Omega$ is the level set of f with $\langle Df(x), \eta(x)\rangle \to +\infty$ as $x \to \partial\Omega$, where η is the outward unit normal to the level sets of f. If*

$$m = \frac{1}{2}\left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$

then $R \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, and $\partial\Omega$ is connected and is a sphere of radius $(2m)^{\frac{1}{n-2}}$.

The proof of Theorem 2 makes use the maximum principles proven in Section 4. The case $n = 3$ or 4 is more subtle because the graphing function of the Schwarzschild solution tends to infinity as $|x| \to \infty$, and difficulty arises when comparing two unbounded graphs. In that case, instead of using the strong maximum principles Theorem 4.3 or Theorem 4.4 directly, we first control the growth at infinity, by the asymptotic flatness of the graph and Theorem 4.6.

Proof of Theorem 2. Suppose that the equality of the Penrose inequality holds. By Corollary 5.3, the scalar curvature of the graph of f is identically zero everywhere and $\partial\Omega$ is a round sphere of radius $(2m)^{\frac{1}{n-2}}$. By translating f, we assume that $f = 0$ on $\partial\Omega$ and $\partial\Omega = S_{(2m)^{\frac{1}{n-2}}} \subset \{x^{n+1} = 0\}$. By Theorem 3, the mean curvature of the graph of f must have a sign. Then by Proposition 2.9 and $H_{\partial\Omega} > 0$ (with respect to inward unit normal), the mean curvature of the graph of f is non-negative everywhere. In particular, this together with the strong maximum principle for the mean curvature equation implies that $\limsup_{|x|\to\infty} f > 0$. Hence, by the assumption of asymptotic flatness, we have either $\lim_{|x|\to\infty} f(x) = C$ for some positive constant C or $\lim_{|x|\to\infty} f(x) = +\infty$.

Let h be the function in Proposition 2.6 which gives the exterior region of the Schwarzschild solution of mass m outside its minimal boundary. By translating h, we assume that its minimal boundary is $S_{(2m)^{\frac{1}{n-2}}} \subset \{x^{n+1} = 0\}$. We consider two cases, depending on the dimension n.

Case 1: $n \geq 5$. In this case, $\lim_{|x|\to\infty} h(x) = C_0$ for some bounded constant C_0.

As explained above, either $\lim_{|x|\to\infty} f(x) = C$ for some bounded constant C or $\lim_{|x|\to\infty} f(x) = +\infty$.

If $C \leq C_0$, we let $u_\lambda = h + \lambda$ for $\lambda \geq 0$. For λ sufficiently large, $u_\lambda > f$. We then continuously decrease λ, until $u_\lambda = f$ at $p \in \mathbb{R}^n \setminus B_{(2m)^{\frac{1}{n-2}}}$ for the first time. If p is an interior point, $u_\lambda \equiv f$ by Theorem 4.3. If p is a boundary point in $S_{(2m)^{\frac{1}{n-2}}}$, $u_\lambda \equiv f$ by Theorem 4.4. Hence, the graph of f is identical to the exterior region of the Schwarzschild solution of mass m outside its minimal boundary.

If $C \geq C_0$ or $\lim_{|x|\to\infty} f(x) = +\infty$, we consider $v_\lambda = h - \lambda$ for $\lambda \geq 0$. Note that $f > v_\lambda$ for λ sufficiently large. We then continuously decrease λ until $f = v_\lambda$ for the first time. Then by either Theorem 4.3 or

Theorem 4.4, we have $f \equiv h$ in $\mathbb{R}^n \setminus B_{(2m)^{\frac{1}{n-2}}}$.

Case 2: $n = 3$ or 4. (In this case, $\lim_{|x|\to\infty} h(x) = \infty$.)

We **claim** that either $\max_{|x|=r} f(x) > h(r)$ or $\max_{|x|=r} f(x) \le h(r)$ for all r sufficiently large. Suppose that the first statement is false. Then there exists a sequence of positive numbers $\{r_k\}$ with $r_k \to \infty$ as $k \to \infty$ so that $\max_{|x|=r_k} f(x) \le h(r_k)$. Then by Theorem 4.6, we have $\max_{|x|=r} f(x) \le h(r)$ for all r sufficiently large. This proves the claim.

Suppose $\max_{|x|=r} f(x) > h(r)$ for all r sufficiently large. By the assumption $\min_{|x|=r} f + C \ge \max_{|x|=r} f(x)$ for r sufficiently large, we have for all $r \ge (2m)^{\frac{1}{n-2}}$

$$\min_{|x|=r} f(x) > h(r) - C',$$

for some constant $C' > 0$. Hence, $f(x) > h(x) - C'$ for all $x \in \mathbb{R}^n \setminus B_{(2m)^{\frac{1}{n-2}}}$. We then continuously decrease C' until $f(x) = h(x) - C'$ for the first time. Then either Theorem 4.3 or Theorem 4.4 implies $f(x) \equiv h(x)$, which leads a contradiction. Hence, $\max_{|x|=r} f(x) \le h(r)$ for all r sufficiently large. Then we can apply the argument as in Case 1 and show that $f(x) \equiv h(x)$. $\square$

Proof of Theorem 4. We argue by contradiction. Suppose there is a complete C^{n+1} hypersurface M of one end with zero scalar curvature in $\mathbb{R}^{n+1}$ which is identical to the Schwarzschild solution h (given in Proposition 2.6) of $m > 0$ outside a compact set. We consider the graph of $h - \lambda$ for some constant $\lambda > 0$. For $\lambda \gg 1$, the graph of $h - \lambda$ has no intersection with M. Then we decrease λ until the graph of $h - \lambda$ touches M at a point p for the first time.

Note that by [12, Theorem 4] the mean curvature of M has a sign. Then by Proposition 2.9 and the fact that the level set of M passing through p is mean convex near p with respect to the inward unit normal, the mean curvature of M near p (with respect to the unit normal vector pointing away from the graph of h) is non-negative. Depending on that p is either an interior point or a boundary point of the graph of $h - \lambda$, we apply either Theorem 4.3 or Theorem 4.4 to conclude that M is identical to the graph of h outside $B_{(2m)^{\frac{1}{n-2}}}$ over $\mathbb{R}^n$. By translation, we may assume that $h = 0$ over $S_{(2m)^{\frac{1}{n-2}}}$. We consider the graph of $-h$ and the region of M near $S_{(2m)^{\frac{1}{n-2}}}$. Note that M is graphical over an open neighborhood of $S_{(2m)^{\frac{1}{n-2}}}$ in $B_{(2m)^{\frac{1}{n-2}}}$ (because the first contact point of the graph of h and M is at the graph of h). Then by applying Theorem 4.5, we obtain a contradiction. □

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