

STATIC POTENTIALS AND AREA MINIMIZING HYPERSURFACES

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ABSTRACT. We show that if an asymptotically flat manifold with horizon boundary admits a global static potential, then the static potential must be zero on the boundary. We also show that if an asymptotically flat manifold with horizon boundary admits an unbounded static potential in the exterior region, then the manifold must contain a complete non-compact area minimizing hypersurface. Some results related to the Riemannian positive mass theorem and Bartnik's quasi-local mass are obtained.

The purpose of this paper is to study the interplay between static potentials and minimal hypersurfaces of an asymptotically flat manifold.

We state the main results. See Appendix A and Appendix B for precise statements of terms used below.

Theorem 1. *Let $n \geq 3$. Let (M, g) be an n -dimensional asymptotically flat manifold with horizon boundary. Suppose (M, g) admits a static potential V . Then V is zero on ∂M .*

As a direct consequence, if V is bounded, then V is either positive or negative everywhere in the interior of M .

The motivation for the above theorem comes from the rigidity of the Riemannian positive mass theorem. In fact, combining with the work of J. Corvino on scalar curvature deformation [12], the work of G. Galloway and P. Miao on static potentials [16, Theorem 4.1], and the rigidity result of O. Chodosh and M. Eichmair [10, Theorem 1.6], the theorem gives another proof to the rigidity of Riemannian positive mass theorem for asymptotically flat manifolds with horizon boundary in three dimensions. We include the proof in Section 3.1.

We also obtain the following generalization of the result of Galloway and Miao [16, Theorem 4.1]. Here, we only assume that the static potential is defined in an exterior region.

Theorem 2. *Let $3 \leq n \leq 7$. Let (M, g) be an n -dimensional asymptotically flat manifold. Suppose the boundary of M is either empty or a disjoint union of smooth minimal hypersurfaces. If one of the asymptotically flat ends admits an unbounded static potential, then there is a complete, non-compact, area minimizing hypersurface in M .*

In the above theorem, we do not assume the scalar curvature of g to be everywhere nonnegative. In the proof, the complete area minimizing hypersurface is obtained as a limit of a sequence of Plateau solutions, and it is a well-known fact that the limiting hypersurface is smooth in dimensions $3 \leq n \leq 7$.

If $n = 3$ and the scalar curvature of g is nonnegative in M , by the result of Chodosh and Eichmair [10, Theorem 1.6], an immediate consequence of Theorem 2 gives the following statement.

The first two authors were partially supported by the NSF through grant DMS 1452477. The third author was partially supported by Simons Foundation Collaboration Grant for Mathematicians #281105.

Corollary 3. *Let (M, g) be a three-dimensional asymptotically flat manifold with horizon boundary. Suppose (M, g) has nonnegative scalar curvature. If the exterior region of (M, g) admits an unbounded static potential, then (M, g) is isometric to Euclidean space.*

We include other results related to Bartnik's quasi-local mass in Section 3.2 and Section 3.3.

Acknowledgements. The first named author is grateful to Jim Isenberg for support. We thank Lucas Ambrozio, Justin Corvino, and Jeff Jauregui for comments on an earlier version of the paper. We also thank the referee for helpful suggestions to improve the presentation.

1. PROOF OF THEOREM 1

To establish the relation between locally area minimizing hypersurfaces and a static potential, we need the following lemma. Recall that in the Appendix B, we define the static potential V as a non-trivial solution to the following static equation

$$-(\Delta V)g + \nabla^2 V - V Ric = 0.$$

Lemma 4 ([22, Equations (9)-(14)]). *Let (Ω, g) be an n -dimensional Riemannian manifold. Suppose that Ω admits a static potential V . Let Σ be a closed, connected, stable minimal hypersurface in Ω . Then we have the following:*

- (1) *Either $V > 0$ or $V < 0$ on Σ , unless V is identically zero on Σ .*
- (2) *Σ is totally geodesic.*

Proof. By the stability inequality, for any $\phi \in C^1(\Sigma)$,

$$\int_{\Sigma} |\nabla_{\Sigma} \phi|^2 d\sigma \geq \int_{\Sigma} (|A|^2 + Ric(\nu, \nu)) \phi^2 d\sigma \geq \int_{\Sigma} Ric(\nu, \nu) \phi^2 d\sigma,$$

where ν is a unit normal vector field to Σ and $d\sigma$ is the $(n-1)$ -volume measure of hypersurfaces. It implies that the first eigenvalue of the operator $\Delta_{\Sigma} + Ric(\nu, \nu)$ is non-positive, where Δ_{Σ} is the induced Laplacian. On the other hand, since Σ is minimal, the restriction of the static potential V on Σ satisfies $\Delta V = \Delta_{\Sigma} V + \nabla^2 V(\nu, \nu)$. By the static equation of V

$$(1.1) \quad 0 = \Delta_{\Sigma} V + \nabla^2 V(\nu, \nu) - \Delta V = \Delta_{\Sigma} V + Ric(\nu, \nu)V.$$

It implies either V is identically zero or V is the first eigenfunction with the zero eigenvalue. If V is zero on Σ , then Σ lies in the zero set of V which is totally geodesic. If V is the first eigenfunction, then V does not vanish on Σ . Substituting V in the stability inequality, we obtain $\int_{\Sigma} |A|^2 V^2 \leq 0$. Thus $|A| \equiv 0$ and Σ is also totally geodesic. □

If, furthermore, Σ is locally area minimizing, a splitting result is obtained by adapting the argument of Galloway in three dimensions [15, Lemma 3]. We note that the argument of Galloway is also extended in [1, Proposition 14 and Appendix B], which covers some of the following results in three dimensions.

Proposition 5. *Let (Ω, g) be an n -dimensional Riemannian manifold with the scalar curvature $R_g = 0$. Suppose that Ω admits a static potential V . Let Σ be a locally area minimizing, closed,*

connected hypersurface in Ω . Suppose V is not identically zero on Σ . Then there is a subset U of Ω and a diffeomorphism $\Phi : \Sigma \times [0, \epsilon) \rightarrow U$ so that the following holds:

- (1) The $(n-1)$ -volume of hypersurfaces $\Sigma_t := \Phi(\Sigma \times \{t\})$ is constant in t .
- (2) The induced scalar curvature R_Σ of Σ_t is zero and V is constant on Σ_t for each t .
- (3) The Ricci curvature of g is zero on U .

Proof. By Lemma 4, we may without loss of generality assume $V > 0$ on Σ . Consider the deformation $\Phi : \Sigma \times [0, \epsilon) \rightarrow \Omega$ given by the normal exponential map with respect to the conformally modified metric $V^{-2}g$ in a collar neighborhood of Σ where $V > 0$. Let $\Sigma_t = \Phi(\Sigma \times \{t\})$ and note $\Sigma_0 = \Sigma$. Let $H(\cdot, t), A(\cdot, t)$ be the mean curvature and second fundamental form of Σ_t in the metric g , respectively. Lemma B.6 implies that $H(\cdot, t) \geq 0$ for $t \in (0, \epsilon)$. From the first variation of area, we have

$$|\Sigma_t| - |\Sigma_0| = \int_0^t \left(- \int_{\Sigma_s} V H(\cdot, s) d\sigma \right) ds.$$

For ϵ sufficiently small, Σ is locally area minimizing. Therefore, the above identity implies that the mean curvature of Σ_t cannot be strictly positive for $t < \epsilon$. Hence $H(\cdot, t) \equiv 0$ and the $(n-1)$ -volume of Σ_t is a constant. By Lemma B.6 again, $A(\cdot, t) \equiv 0$ and Σ_t is totally geodesic for $t \in [0, \epsilon)$ with respect to the metric g .

Furthermore, using the first variation of the second fundamental form (see, for example, [10, p. 993] and the references therein), we obtain, for vectors X, Y tangential to Σ_t ,

$$\nabla_\Sigma^2 V(X, Y) + Rm(\nu, X, Y, \nu)V = 0,$$

where ∇_Σ denotes the connection of Σ_t , ν is a unit normal vector to Σ_t (both with respect to the metric g), and Rm is the Riemann curvature tensor of (Ω, g) (with the sign convention that the Ricci tensor is the trace on the first and fourth components of Rm). Because Σ_t is totally geodesic, $\nabla_\Sigma^2 V(X, Y) = \nabla^2 V(X, Y)$ for tangential vectors X, Y . Then by the static equation (B.1), the assumption that $R_g = 0$, and $V > 0$, we obtain $Ric(X, Y) = -Rm(\nu, X, Y, \nu)$. For an orthonormal frame $\{E_i\}$ on Σ_t ,

$$\begin{aligned} Ric(X, Y) &= Rm(\nu, X, Y, \nu) + \sum_i Rm(E_i, X, Y, E_i) \\ &= -Ric(X, Y) + Ric_\Sigma(X, Y), \end{aligned}$$

where we also use the Gauss equation in the second equality and denote by Ric_Σ the Ricci tensor of Σ_t induced from g . It gives that, for all tangential vector fields X, Y to Σ_t ,

$$(1.2) \quad Ric(X, Y) = \frac{1}{2} Ric_\Sigma(X, Y)$$

and hence, combining the previous formulas gives

$$(1.3) \quad \nabla_\Sigma^2 V = \frac{1}{2} V Ric_\Sigma$$

$$(1.4) \quad \Delta_\Sigma V = \frac{1}{2} V R_\Sigma,$$

where R_Σ denotes the scalar curvature of Σ_t . Take the divergence of (1.3) on Σ_t and note $\operatorname{div}_\Sigma(\nabla_\Sigma^2 V) = d(\Delta_\Sigma V) + Ric_\Sigma \cdot \nabla_\Sigma V$, where the dot in the last term denotes tensor contraction.

Hence, we derive that, on each Σ_t ,

$$\begin{aligned}
0 &= d(\Delta_\Sigma V) + Ric_\Sigma \cdot \nabla_\Sigma V - \frac{1}{2} \left(\frac{1}{2} V dR_\Sigma + Ric_\Sigma \cdot \nabla_\Sigma V \right) \\
&= \frac{1}{2} d(R_\Sigma V) + \frac{1}{2} Ric_\Sigma \cdot \nabla_\Sigma V - \frac{1}{4} V dR_\Sigma \\
&= \frac{1}{4} V dR_\Sigma + \frac{1}{2} R_\Sigma dV + V^{-1} \nabla_\Sigma^2 V \cdot \nabla_\Sigma V \\
&= \frac{1}{4} V^{-1} d(R_\Sigma V^2 + 2|\nabla_\Sigma V|^2).
\end{aligned}$$

This implies that $R_\Sigma V^2 + 2|\nabla_\Sigma V|^2$ is constant on each Σ_t and in fact, by (1.4),

$$R_\Sigma V^2 + 2|\nabla_\Sigma V|^2 = 0.$$

It gives $R_\Sigma \leq 0$. On the other hand, by (1.4),

$$\int_{\Sigma_t} R_\Sigma d\sigma = 2 \int_{\Sigma_t} V^{-2} |\nabla_\Sigma V|^2 d\sigma \geq 0.$$

Hence $R_\Sigma = 0$ and V is constant on Σ_t for each $t \in [0, \epsilon)$. By (1.3), $Ric_\Sigma = 0$ and by (1.2) $Ric(X, Y) = 0$ for vectors tangential to Σ_t . By the Codazzi equation, $Ric(X, \nu)$ is zero, and by the Gauss equation, $Ric(\nu, \nu)$ is zero. Thus, the Ricci tensor is zero in U . \square

Proof of Theorem 1. Note that the scalar curvature of g is constant on M and hence must be zero, by asymptotic flatness. If V is not zero on ∂M , by Proposition 5, a collar neighborhood of ∂M in M splits as a foliation of minimal hypersurfaces. It contradicts that M contains no closed minimal hypersurfaces other than ∂M . We also note that since V is not identically zero, each component of the zero set of V is a regular hypersurface, and hence ∂M is itself a connected component of the zero set.

For the rest of the proof, we assume V is bounded. By Proposition B.4, V has the following expansion on each end N_k , for a nonzero constant A_k ,

$$V(x) = A_k + O(|x|^{2-n}).$$

We may assume $A_1 > 0$ (otherwise, consider $-V$). It implies that $A_k > 0$ for all other k ; otherwise, the zero set of M is nonempty in the interior M , which would imply that M has a closed minimal hypersurface other than ∂M . Therefore, by the strong maximum principle for harmonic functions, $V > 0$ in M . \square

We remark that in the preceding proof, we can further apply the Hopf boundary point lemma to conclude that, for $V > 0$ in M , the normal derivative $\nabla_\nu V > 0$ on ∂M with respect to the normal vector ν to ∂M pointing into M .

2. PROOF OF THEOREM 2

The following observation is due to G. Galloway and P. Miao in [16]. We recall that B_r denotes the large coordinate ball of radius r with respect to the chart at infinity, as defined in Appendix A.

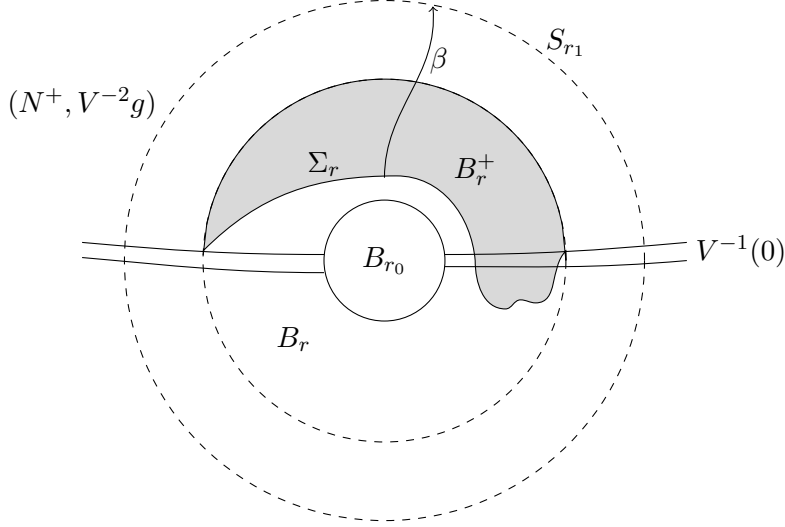


FIGURE 1. An illustration for the contradictory argument in the proofs of Theorem 2 and Lemma 6. If the Plateau solution Σ_r was disjoint from B_{r_0} , it would have separated B_{r_0} from infinity in N^+ . Then a minimizing geodesic β in the interior with respect the modified metric $V^{-2}g$ from Σ_r to a large coordinate sphere S_{r_1} exists, which leads to a contradiction.

Lemma 6 (Essentially [16, Theorem 3.1]). *Let $n \geq 3$ and let (M, g) be an n -dimensional asymptotically flat manifold. Let N be one of the ends. Suppose $N \setminus B_{r_0}$ admits a static potential V for some $r_0 > 0$. Let N^+ be an unbounded component of the complement of $V^{-1}(0)$ in $N \setminus B_{r_0}$. Let Σ be a compact two-sided minimal hypersurface in N^+ with boundary either on $V^{-1}(0)$ or empty. Then Σ cannot separate B_{r_0} from infinity in N^+ ; that is, every unbounded component of the complement of Σ in $N^+ \cup B_{r_0}$ contains B_{r_0} .*

Proof. Suppose on the contrary that there is an unbounded component Ω of the complement of Σ in $N^+ \cup B_{r_0}$ that does not contain B_{r_0} . Note that V is globally defined and nonzero on Ω , and $\partial\Omega$ consists of Σ and a subset of $V^{-1}(0)$. We may assume $V > 0$ in Ω ; otherwise consider $-V$. We shall consider geodesics in the modified metric $\bar{g} = V^{-2}g$ emitting from Σ into Ω . It is shown in [16, Lemma 3.1] that those geodesics cannot reach the zero set of V in the finite \bar{g} length, and any two disjoint points in the zero set have infinite \bar{g} distance. The rest of the argument follows from [16, Theorem 3.1] which we briefly summarize below. Consider a large coordinate sphere S_r that intersects Ω and is disjoint from $\Sigma \cap \Omega$. There exists a *minimizing* geodesic in the modified metric \bar{g} emitting from the interior of Σ in Ω that reaches $S_r \cap \Omega$. In a tubular neighborhood of the geodesic, we consider the level set of the distance function with respect to the \bar{g} metric from Σ . By the monotonicity formula Lemma B.6, those hypersurfaces have nonpositive mean curvature in the metric g (with respect to $-\nu$, where ν is as in Lemma B.6). It ultimately leads a contradiction to the convexity of large coordinate spheres and the maximum principle. \square

Proof of Theorem 2. Let V be an unbounded static potential on one of the ends, say N . By Proposition B.4, V is asymptotic to a linear combination of coordinate functions on the end N . By rotating the asymptotically flat coordinate chart of N and rescaling V if necessary, we assume

$V(x)$ is asymptotic to x^n . By Lemma B.5, there is r_0 sufficiently large so that each component of $V^{-1}(0)$ is a graph $x^n = f(x^1, \dots, x^{n-1})$ that intersects with S_r in a nearly equatorial $(n-2)$ sphere for $r > r_0$. We may assume r_0 sufficiently large so that S_{r_0} does not intersect any closed minimal hypersurfaces.

For $r > r_0$, consider the orientable Plateau solution Σ_r whose boundary spans the intersection of S_r and a component of $V^{-1}(0)$. We claim that Σ_r must intersect B_{r_0} for all $r > r_0$. Suppose on the contrary that Σ_r is disjoint from B_{r_0} . Since Σ_r separates B_r , there is a component B_r^+ of the complement of Σ_r in B_r that does not contain B_{r_0} . We may without loss of generality assume that B_r^+ contains the top portion of S_r (otherwise, consider $-V$). Then we consider the *top* component N^+ of the complement of $V^{-1}(0)$ in $N \setminus B_{r_0}$, i.e. the component containing all points with sufficiently large x^n -coordinate values. Notice that $\Sigma_r \cap N^+$ separates B_{r_0} from infinity in N^+ , as the complement of Σ_r in $N^+ \cup B_{r_0}$ has only one unbounded component Ω , and Ω cannot intersect the components of $B_r \setminus \Sigma_r$ other than B_r^+ , by connectedness of Ω . It gives a contradiction to Lemma 6.

Since Σ_r intersects S_{r_0} for all $r > r_0$ and $\{\Sigma_r\}$ has a uniformly local area bound, by standard geometric measure theory, a subsequence of Σ_r converges to a nonempty complete area minimizing hypersurface Σ intersecting S_{r_0} as $r \rightarrow \infty$. Since S_{r_0} does not intersect any closed minimal hypersurface, Σ is unbounded. \square

3. APPLICATIONS

3.1. Rigidity of the Riemannian positive mass theorem. The Riemannian positive mass theorem is due to R. Schoen and S. T. Yau [27, 28, 29]. Other proofs for the three-dimensional case can be found in [18] and [20].

Here we use static potentials to give another proof of the rigidity of the Riemannian positive mass theorem in three dimensions. The argument for complete manifolds without boundary may have already been known to the experts. Here we use Theorem 1 and extend the argument to asymptotically flat manifolds with horizon boundary.

Theorem 7. *Let (M, g) be a three-dimensional asymptotically flat manifold with horizon boundary and nonnegative scalar curvature. Assume $g \in C_{loc}^{4,\alpha}$. If one of the ends has zero ADM mass, then (M, g) is isometric to Euclidean space.*

Proof. We first apply the argument similar to [12, Theorem 8] to show that every precompact open subset Ω in M admits a static potential $V \in C^{4,\alpha}(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$. The only difference is that in order to keep the minimal boundary condition, we consider the conformal Laplacian with the *Neumann boundary condition* as follows. Suppose on contrary there is a precompact open subset Ω which does not admit a static potential. By [12, Theorem 1] (see, also, Theorem B.2), there is a $C_{loc}^{2,\alpha}$ metric \bar{g} with positive scalar curvature in Ω such that \bar{g} coincides with g outside Ω . Then [27, Lemma 3.3] implies there exists a unique positive solution u to $\Delta_{\bar{g}}u - \frac{1}{8}R_{\bar{g}}u = 0$ in M with $\frac{\partial u}{\partial \nu} = 0$ on ∂M and $u(x) \rightarrow 1$ as $|x| \rightarrow \infty$ so that $u^4\bar{g}$ has negative mass. It gives a contradiction to the positive mass inequality.

Next we show that there is a global vacuum static potential $V \in C_{loc}^{4,\alpha}(M)$. Let B_k be an exhaustion sequence of coordinate balls of M . As shown in the previous paragraph, each B_k

admits a static potential $V_k \in C^{4,\alpha}(B_k) \cap C^{2,\alpha}(\overline{B_k})$. For a fixed $r_0 > 1$, we may normalize V_k so that $\max_{S_{r_0} \cup \partial M} (|V_k| + |\nabla V_k|) = 1$ for all $k > r_0$. Then by maximum principle (for V_k in B_{r_0}) and by Proposition B.4 (for V_k in the annulus $B_r \setminus B_{r_0}$), we have $\sup_{B_r} |V_k| \leq Cr$ uniformly for all $k > r$. Thus, $\|V_k\|_{C^{2,\alpha}(B_r)}$ is bounded uniformly in k by the Schauder estimate. By Arzela-Ascoli theorem and taking the diagonal sequence, we obtain a $C_{\text{loc}}^{2,\frac{\alpha}{2}}$ limiting function V in M with $\sup_{S_{r_0} \cup \partial M} (|V| + |\nabla V|) = 1$. Hence, V is a nontrivial solution to the static equation, and $V \in C_{\text{loc}}^{4,\alpha}(M)$ by elliptic regularity.

If V is unbounded in M , then M has a complete, non-compact, area minimizing surface by the result of Galloway and Miao [16, Theorem 4.1] (or Theorem 2). The rigidity follows from the work of Chodosh and Eichmair [10, Theorem 1.6].

We now discuss the case that V is bounded. Note that since any two ends must be separated by a minimal surface and M does not contain any minimal surfaces in its interior, M has only one end with zero ADM mass. If V is bounded, by Proposition B.4, V goes to a constant A at infinity. More specifically,

$$V = A + o(|x|^{2-n}).$$

Integrating $V\Delta V = 0$ yields

$$\begin{aligned} 0 &= \int_M V\Delta V \, d\mu = - \int_M |\nabla V|^2 \, d\mu + \lim_{r \rightarrow \infty} \int_{S_r} V \frac{\partial V}{\partial r} \, d\sigma + \int_{\partial M} V \frac{\partial V}{\partial r} \, d\sigma \\ &= - \int_M |\nabla V|^2 \, d\mu, \end{aligned}$$

where we use that if ∂M is nonempty, then V is zero on ∂M by Theorem 1. We obtain $|\nabla V| = 0$ in M . Hence V is constant on M and then g is Ricci flat by the static equation (B.1). For three-dimensional manifolds, it implies g has zero sectional curvature and hence (M, g) must be isometric to Euclidean space. □

We remark that Theorem 7 and Theorem 1 are closely related to the uniqueness of static black holes, which says that an asymptotically flat manifold admitting a global static potential $V \geq 0$ must be isometric to a Schwarzschild metric. However, we emphasize that our proofs to Theorem 7 and Theorem 1 are *independent* of the uniqueness of static black holes. The proof of Bunting and Masood-ul-Alam [9] and the later extensions in [11, 22] *use* Theorem 7. Although some results are obtained independent of Theorem 7 in [19, 25, 26], more stringent conditions, such as positivity of V in the interior of M , $V = 0$ on ∂M , and connectedness of ∂M , are assumed.

3.2. The mass minimizer of Bartnik's quasi-local mass. We recall the definition of Bartnik's quasi-local mass proposed by R. Bartnik [4] and revised by H. Bray [6] (see also [7]) as follows.

Definition 8 ([6, 7]). *Let (N, h) be a complete, asymptotically flat three-manifold with nonnegative scalar curvature. Let $\Omega \subset N$ be a bounded subset such that $\partial\Omega$ is outer-minimizing in (N, h) . Let \mathcal{PM} be the set of complete, asymptotically flat 3-manifolds (M, g) with nonnegative scalar curvature so that (Ω, h) isometrically embeds in (M, g) and $\partial\Omega$ is outer-minimizing in (M, g) . The Bartnik quasi-local mass is defined as*

$$\mathfrak{m}_B(\Omega, h) = \inf \{m(M, g) : (M, g) \in \mathcal{PM}\},$$

where we recall that $m(M, g)$ is the ADM mass of (M, g) .

In this definition, the outer-minimizing assumption of $\partial\Omega$ is imposed so that the Hawking mass of $\partial\Omega$ gives a lower bound of $\mathfrak{m}_B(\Omega, h)$. This follows from the inverse mean curvature flow argument of Huisken and Ilmanen in the proof of the Riemannian positive mass theorem [18].

Example 9. Let $(N, h) \in \mathcal{PM}$ be rotationally symmetric such that the scalar curvature of h is identically zero outside a bounded, rotationally symmetric subset Ω . (Such an (N, h) can be easily constructed by an ODE method.) By uniqueness of rotationally symmetric solutions of the vacuum Einstein equations (or simply solving the ODE of the zero scalar curvature equation), $(N \setminus \Omega, h)$ is isometric to an exterior region of a Schwarzschild manifold. In particular, $\partial\Omega$ is strictly outer-minimizing in (N, h) and $m(N, h)$ equals the Hawking mass of $\partial\Omega$. As a result, $m(N, h) = \mathfrak{m}_B(\Omega, h)$. Using the well-known facts about Schwarzschild manifolds, we see that this mass minimizer (N, h) admits a static potential V in the exterior region $N \setminus \Omega$, and V approaches a constant at infinity. \square

Next we show that the above assertion on static potentials holds in general. Namely, if a suitable mass minimizer exists in \mathcal{PM} , then the exterior region of the mass minimizer admits a static potential that goes to a constant at infinity.

Theorem 10. *Let $\Omega \subset (N, h)$ be a bounded subset where $(N, h) \in \mathcal{PM}$. Suppose there exists $(M, g) \in \mathcal{PM}$ such that $\partial\Omega$ is strictly outer-minimizing in (M, g) and $\mathfrak{m}_B(\Omega, h) = m(M, g)$, then $(M \setminus \overline{\Omega}, g)$ admits a static potential that goes to a constant at infinity.*

Remark 11. An analogous result is also obtained by M. Anderson and J. Jauregui [2, Theorem 1.1] using a different approach. Though, note that their definition of Bartnik's quasi-local mass is slightly different from ours because the minimization in their definition is taken over a large class of asymptotically flat 3-manifolds. The first named author is very grateful to Jeff Jauregui for kindly explaining their proof.

Proof. Using the mass minimizing property of (M, g) , a recent result of J. Corvino [13, Corollary 1.2] shows that $(M \setminus \overline{\Omega}, g)$ admits a static potential V . (The strictly outer-minimizing assumption on $\partial\Omega$ guarantees that the competitors produced in [13, Corollary 1.2] still lie in \mathcal{PM} .) By asymptotics of static potentials, either V goes to a constant or V is unbounded (see [5] and [23], or Proposition B.4 below). If V goes to a constant, the claim follows.

Now we assume that V is unbounded. Theorem 2 implies that there is a complete, non-compact, area minimizing surface in (M, g) . We then invoke the rigidity result of Chodosh and Eichmair in [10, Theorem 1.6] to conclude that (M, g) is isometric to Euclidean space. Then it is obvious that the constant function is a static potential on Euclidean space. \square

3.3. Geometric properties of a static extension. Given a Riemannian metric γ and a function H on a 2-sphere, we say that an asymptotically flat 3-manifold (M, g) with boundary $\Sigma = \partial M$ is a *static extension subject to the boundary data* (γ, H) if

- (1) Σ is diffeomorphic to the 2-sphere, and the induced metric from g on Σ is isometric to γ .
- (2) The mean curvature of Σ with respect to the unit normal vector on Σ pointing into (M, g) is given by H .

(3) (M, g) admits a bounded static potential.

Below we give the sufficient conditions on (γ, H) so that the static extension has no closed minimal surfaces that locally minimize the area.

Theorem 12. *Suppose the pair (γ, H) satisfies*

$$H > 0 \quad \text{and} \quad K_\gamma \geq \frac{1}{4}H^2,$$

where K_γ denotes the Gauss curvature of γ . Then any static extension (M, g) subject to the boundary data (γ, H) does not have closed, locally area minimizing surfaces.

Proof. Let V be a bounded static potential on (M, g) . By Proposition B.4 and normalizing, we may assume $V \rightarrow 1$ at infinity.

We now use the argument in [22, Proposition 3] to show that $V > 0$ in M . By the static equation and recall $R_g = 0$, we have the following identity on Σ :

$$\begin{aligned} (3.1) \quad 0 = \Delta V &= \Delta_\Sigma V + H \frac{\partial V}{\partial \nu} + \nabla^2 V(\nu, \nu) \\ &= \Delta_\Sigma V + H \frac{\partial V}{\partial \nu} + Ric(\nu, \nu)V, \end{aligned}$$

where ν is the unit normal vector on Σ pointing into M . By the Gauss equation,

$$Ric(\nu, \nu) = \frac{1}{2} (H^2 - |A|^2 - 2K_\gamma),$$

where A denotes the second fundamental form of Σ . Combining the above identities gives

$$(3.2) \quad \Delta_\Sigma V + H \frac{\partial V}{\partial \nu} + \frac{1}{2} (H^2 - |A|^2 - 2K_\gamma) V = 0.$$

Because V is harmonic in M , by maximum principle and $V \rightarrow 1$ at infinity, we may assume that $\inf_M V$ occurs on Σ and V is not a constant. Otherwise the claim $V > 0$ follows easily. Let $V(y) = \inf_M V$ for some $y \in \Sigma$. Using the Hopf boundary point lemma and noting $V(y) = \min_\Sigma V$, we have the following inequalities at y ,

$$\frac{\partial}{\partial \nu} V(y) > 0 \quad \text{and} \quad \Delta_\Sigma V(y) \geq 0.$$

On the other hand, the assumption on H and K_γ implies that

$$\frac{1}{2} (H^2 - |A|^2 - 2K_\gamma) \leq \frac{1}{4}H^2 - K_\gamma \leq 0,$$

Combing the above inequities and (3.2), we conclude that $V(y) > 0$ and hence $V > 0$ in M .

Suppose, to give a contradiction, that there is a closed, locally area minimizing surface in M . By Proposition 5, g must be Ricci flat in an open neighborhood of the minimal surface. Since $V > 0$ and g is static, g is analytic on M (cf. [12]). Hence, (M, g) has vanishing Ricci curvature. In three dimensions, this implies (M, g) is isometric to an exterior region in the Euclidean space, which is free of closed minimal surfaces. It gives a contradiction. \square

APPENDIX A. ASYMPTOTICALLY FLAT MANIFOLDS

Let $n \geq 3$. An n -dimensional (connected) manifold (M, g) is said to be *asymptotically flat* if $M \setminus K = \bigcup_k N_k$ for some compact subset $K \subset M$ and, for $q > \frac{n-2}{2}$, there is a coordinate chart on each end

$$N_k \cong \mathbb{R}^n \setminus B_1(0)$$

so that the components of the metric tensor satisfy

$$|g_{ij} - \delta_{ij}| + |x| |\partial_k g_{ij}| + |x|^2 |\partial_k \partial_\ell g_{ij}| \leq C |x|^{-q}.$$

We also assume the scalar curvature R_g is integrable in M and $g \in C_{\text{loc}}^{2,\alpha}(M)$.

For $r > 1$, we let $B_r = \bigcup_{x \in N_k} \{|x| \leq r\} \cup K$ be the closed coordinate ball with respect to the above charts, and let the coordinate sphere $S_r = \bigcup_{x \in N_k} \{|x| = r\}$.

Throughout this note, we follow the convention that stable minimal hypersurfaces are two-sided. We say that M has a *horizon boundary* if the boundary ∂M , possibly empty, is a disjoint union of smooth closed minimal hypersurfaces, M contains no other closed minimal hypersurfaces, and we further assume that ∂M is locally area minimizing if $n \geq 8$. (Note that if $3 \leq n \leq 7$, ∂M is area minimizing, implied by other two conditions.)

A complete non-compact hypersurface Σ in M is said to be *area minimizing* if $\Sigma \cap B_r$ is a Plateau solution with the boundary spanning $\Sigma \cap S_r$ for all r sufficiently large.

We define the ADM mass of (M, g) by

$$m = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r \rightarrow \infty} \int_{|x|=r} \sum_{i,j=1}^n (g_{ij,i} - g_{ii,j}) \nu^j d\sigma_0,$$

where $d\sigma_0$ is the $(n-1)$ -volume measure induced from ambient Euclidean metric. We may write $m(M, g)$ to emphasize the dependence on the asymptotically flat manifold (M, g) .

APPENDIX B. STATIC POTENTIAL

Let (Ω, g) be an n -dimensional Riemannian manifold. Let $L_g^* : H_{\text{loc}}^2(\Omega) \rightarrow L_{\text{loc}}^2(\Omega)$ be a differential operator defined by

$$L_g^* V = -(\Delta V)g + \nabla^2 V - V Ric,$$

where ∇^2 is the Hessian operator and Ric is the Ricci tensor of g . A *static potential* V is a scalar valued function on Ω that satisfies $L_g^* V = 0$ and is not identically zero. The equation $L_g^* V = 0$ is equivalent to the following equation

$$(B.1) \quad \nabla^2 V = \left(Ric - \frac{1}{n-1} R_g g \right) V.$$

By elliptic regularity, if $g \in C_{\text{loc}}^{k,\alpha}$ for some $k \geq 2$, then a static potential V is $C_{\text{loc}}^{k,\alpha}(\Omega)$, and $V \in C^{k-2,\alpha}(\bar{\Omega})$ if $\bar{\Omega}$ is bounded (see, e.g. [14, Proposition 2.1] and letting $X = 0$ there). We say that (Ω, g) *admits a static potential* if there is a static potential V defined on Ω .

Lemma B.1 ([12, Proposition 2.3 and Proposition 2.6]). *Let (Ω, g) be a connected manifold admitting a static potential V . Then*

- (1) *The scalar curvature of g is constant on Ω .*

(2) The zero set of V is a totally geodesic regular hypersurface in Ω .

A static potential appears to be the only obstruction to promoting scalar curvature locally. The following statement is a special case of [12, Theorem 1]. Also see, e.g. [?].

Theorem B.2 (See [12, Theorem 1]). *Let Ω is a bounded open subset of a Riemannian manifold (M, g) and $g \in C^{4,\alpha}(\bar{\Omega})$. Suppose $\partial\Omega$ is smooth. If Ω does not admit a static potential, then there is a metric $\bar{g} \in C_{\text{loc}}^{2,\alpha}(M)$ so that $\bar{g} = g$ outside Ω and $R(\bar{g}) > R(g)$ in Ω .*

To analyze the asymptotics of a static potential, we need the following ODE lemma. In this paper, we only apply the case that $Z(t)$ is real-valued, but for other future applications toward the system of Einstein constraint equations, we include the following general statement.

Lemma B.3. *Let $n \geq 1$. Let $Z : [1, \infty) \rightarrow \mathbb{R}^k$ be a C^2 vector valued function satisfying the differential equation*

$$Z''(t) = A(t)Z' + B(t)Z(t),$$

where $A(t), B(t)$ are continuous $k \times k$ matrix functions on $[1, \infty)$ satisfying $|A(t)| + t|B(t)| \leq C_1 t^{-1-q}$ for some constants $C_1 > 0$ and $q \geq 0$. Then $|Z| + t|Z'| \leq C_2 t$ where C_2 depends only on C_1 and $Z(1), Z'(1)$. Furthermore, if Z vanishes to infinite order at infinity, i.e., for each $N > 0$ there is a constant c_N such that $|Z(t)| \leq c_N t^{-N}$ on $[1, \infty)$, then Z is identically zero on $[1, \infty)$.

Proof. Define the function $h = t^2|Z'|^2 + |Z|^2 \geq 0$. Applying uniqueness for ODE, if $h(t) = 0$ for some value of t , then h is identically zero on $[1, \infty)$, so we may assume that $h > 0$ everywhere. Compute

$$h' = 2t|Z'|^2 + 2t^2 Z' \cdot Z'' + 2Z \cdot Z'.$$

Using the equation for Z'' and the bound on the coefficients, we obtain that $|h'| \leq \frac{3(1+C_1)}{t}h$. Denote by $2a = 3(1 + C_1)$. Solving the differential inequality yields

$$h(1)t^{-2a} \leq h(t) \leq h(1)t^{2a}.$$

The lower bound implies that any nontrivial solution Z cannot vanish to infinite order at infinity.

The differential equation of Z'' implies that

$$|Z''(t)| \leq t^{-1} \sqrt{2h(t)} (|A(t)| + t|B(t)|).$$

Hence, $|Z''(t)| \leq C_1 \sqrt{2h(1)} t^{-2-q+a}$. By integration, we have

$$|Z'(t)| \leq |Z'(1)| + t \max_{[1,t]} |Z''| \leq |Z'(1)| + C_1 \sqrt{2h(1)} t^{-1-q+a}$$

$$|Z(t)| \leq |Z(1)| + t \max_{[1,t]} |Z'| \leq |Z(1)| + t|Z'(1)| + C_1 \sqrt{2h(1)} t^{-q+a}$$

and inserting these into the definition of $h(t)$ we find

$$h(t) \leq 3(Z(1))^2 + 5t^2|Z'(1)|^2 + 7C_1^2 h(1) t^{-2q+2a}.$$

It implies that the growth rate of h can be further improved by a bootstrap argument, until the highest power of t is quadratic. Thus, $|Z| + t|Z'| \leq C_2 t$ for some constant C_2 depending only on $C_1, Z(1), Z'(1)$.

□

Most of the statement in Proposition B.4 below is known and can be found in [5, Appendix C] and [23]. We include the statement and the arguments here because it seems that the estimate (B.2) below used in the proof of Theorem 7 is not explicitly stated in the literature.

Proposition B.4. *Let (M, g) be an n -dimensional asymptotically flat manifold. Let N be one of the ends. Suppose $N \cap (B_{r_1} \setminus B_{r_0})$ admits a static potential V for some $1 < r_0 < r_1$. Then V is at most of linear growth, in the sense that there is a constant C , depending only on $\max_{S_{r_0}}(|V| + |\nabla V|)$ and (M, g) , such that for each $r \in (r_0, r_1)$*

$$(B.2) \quad \sup_{B_r \setminus B_{r_0}} |V| \leq Cr.$$

Furthermore, if V is defined on all of N , then either one of the following properties holds on the end N :

- (1) V is identically zero.
- (2) $V = \sum_{i=1}^n a_i x^i + O^{2,\alpha}(1 + |x|^{1-q} \log |x|)$ for some constants a_1, \dots, a_n , not all zero.
- (3) $V = a_0 - a_0 m |x|^{2-n} + O^{2,\alpha}(|x|^{1-n} + |x|^{2-n-q} \log |x|)$, where a_0 is a nonzero constant and m is the ADM mass of the end N .

Proof. We compute with respect to the polar coordinate chart of $\{x\}$

$$\begin{aligned} V_r &= \sum_i \frac{x^i}{r} \frac{\partial V}{\partial x^i} \\ V_{rr} &= \sum_{i,j} \frac{\partial^2 V}{\partial x^i \partial x^j} \frac{x^i x^j}{r^2} = \sum_{i,j} \nabla^2 V(\partial x^i, \partial x^j) \frac{x^i x^j}{r^2} + \sum_{i,j,k} \Gamma_{ij}^k \frac{x^i x^j}{r^2} \frac{\partial}{\partial x^k} V. \end{aligned}$$

By the static equation, V satisfies a differential equation of the form in Lemma B.3 along each fixed angular direction, and thus $|V| + r|V_r| + r^{2+q}|V_{rr}| \leq Cr$ where C depends only on the asymptotically flat metric g and the values of V, V_r on S_{r_0} . By compactness of S_{r_0} , the constant C can be chosen uniformly among the angular directions. Therefore, we have

$$|V| + |x||\partial V| + |x|^{2+q}|\partial^2 V| \leq C|x|,$$

which, in particular, proves the first assertion.

From now on, we assume that V is defined on N . Since V is harmonic, by the growth rate bound and harmonic expansions (e.g. [3, Theorem 1.17] and [21]), V is asymptotic to a harmonic function of homogeneous degree at most one:

$$V(x) = \sum_i a_i x^i + O^{2,\alpha}(1 + |x|^{1-q} \log |x|)$$

for some constants a_1, \dots, a_n . If the constants are all zero, then again by the harmonic expansion, there are constants a_0, b such that $V(x) = a_0 + b|x|^{2-n} + O^{2,\alpha}(|x|^{1-n} + |x|^{2-n-q} \log |x|)$. Compute

$$(\nabla^2 V)_{ij} = \frac{\partial^2 V}{\partial x^i \partial x^j} - \sum_k \Gamma_{ij}^k \frac{\partial V}{\partial x^k} = -\frac{(n-2)b\delta_{ij}}{|x|^n} + \frac{n(n-2)bx^i x^j}{|x|^{n+2}} + O^{0,\alpha}(|x|^{-1-n} + |x|^{-n-q} \log |x|).$$

Using the alternative definition of the ADM mass (see, e.g. [24] and also [17, Equation 1.4] for $n = 3$) and the static equation

$$a_0 m = \frac{1}{(n-1)(2-n)\omega_{n-1}} \lim_{r \rightarrow \infty} \int_{\{|x|=r\} \cap E} V R_{ij} x^i \frac{x^j}{|x|} d\sigma = -b.$$

It gives the desired expansion.

If a_0, a_1, \dots, a_n are all zero, then V goes to zero at infinity. Applying the static equation and bootstrapping yields that V vanishes to infinite order at infinity. Applying Lemma B.3 to the differential equation of V along r implies that V is identically zero. □

Lemma B.5. *Let (M, g) be an n -dimensional asymptotically flat manifold. Let N be one of the ends and suppose N admits a static potential V with the asymptotics $V(x) = x^n + o(|x|)$ as $|x| \rightarrow \infty$. Then there is $r_0 > 0$ large such that each component Σ of $V^{-1}(0)$ in $N \setminus B_{r_0}$ is given by a graph $x^n = f(x^1, \dots, x^{n-1})$ and Σ intersects S_r transversely in a nearly equatorial $(n-2)$ sphere for $r > r_0$.*

Proof. By the previous proposition, we have $\nabla_{\partial x^n} V = 1 + O(|x|^{\gamma-1}) > 0$ for $|x|$ large, where $\max\{1 - q, 0\} < \gamma < 1$. Let $x' = (x^1, \dots, x^{n-1})$. Then by the implicit function theorem, each component of the zero set is given by a graph $x^n = f(x')$ with $|\nabla f| \leq C|x'|^{\gamma-1}$. Then $V(x', f(x')) = 0$ implies that $|f(x')| \leq C|x'|^\gamma$. The constant C above can be chosen uniform for all components. If r_0 sufficiently large, each component of $V^{-1}(0)$ intersects S_r transversely near the equator, for all $r > r_0$. □

We include the following monotonicity formula of G. Galloway [15], which is a key geometric ingredient in the proofs of Theorem 1 and Theorem 2. Let (Ω, g) be an n -dimensional Riemannian manifold that admits a static potential. Let Σ be a two-sided smooth hypersurface in Ω . If $V > 0$ in Ω , let $\Phi : \Sigma \times [0, \epsilon) \rightarrow \Omega$ be the normal exponential map with respect to the conformally modified metric $\bar{g} = V^{-2}g$. In particular, $\Phi(x, 0) = x$ and

$$\left. \frac{\partial}{\partial t} \Phi(x, t) \right|_{t=0} = V(x)\nu(x),$$

where ν is the unit normal vector in the metric g . Let $\Sigma_t = \Phi(\Sigma \times \{t\})$ and let $H(x, t), A(x, t)$ be the mean curvature and second fundamental form of $x \in \Sigma_t$ with respect to ν in the metric g .

Lemma B.6 (Monotonicity formula [15, Lemma 3], see also [8, Proposition 3.2]). *The mean curvature and second fundamental form of Σ_t satisfy the following differential equality*

$$\frac{d}{dt} \left(\frac{H}{V} \right) = |A|^2.$$

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